Permanental fields, loop soups and continuous additive functionals

Yves Le Jan   Michael B. Marcus   Jay Rosen *

September 11, 2012

Abstract

A permanental field, $\psi = \{\psi(\nu), \nu \in \mathcal{V}\}$, is a particular stochastic process indexed by a space of measures on a set S. It is determined by a kernel $u(x,y)$, $x, y \in S$, that need not be symmetric and is allowed to be infinite on the diagonal. We show that these fields exist when $u(x,y)$ is a potential density of a transient Markov process X in S.

A permanental field ψ can be realized as the limit of a renormalized sum of continuous additive functionals determined by a loop soup of X, which we carefully construct. A Dynkin type isomorphism theorem is obtained that relates ψ to continuous additive functionals of X (continuous in t), $L = \{L^\nu_t, (\nu, t) \in \mathcal{V} \times R_+\}$. It is used to obtain sufficient conditions for the joint continuity of L, that is, that L is continuous on $\mathcal{V} \times R_+$. The metric on $\mathcal{V}$ is given by a **proper norm**.

1 Introduction

In [16] we use a version of the Dynkin isomorphism theorem to analyze families of continuous additive functionals of symmetric Markov processes in terms of associated second order Gaussian chaoses that are constructed from Gaussian fields with covariance kernels that are the potential densities of the symmetric Markov processes.

*Research of the last two authors was supported by grants from the National Science Foundation and PSCCUNY. In addition, research of the second author was partially supported by grant number 208494 from the Simons Foundation.

[0] Key words and phrases: permanental fields, Markov processes, 'loop soups', continuous additive functionals.

[0] AMS 2000 subject classification: Primary 60K99, 60J55; Secondary 60G17.

In this paper we define a permanental field, $\psi = \{\psi(\nu), \nu \in \mathcal{V}\}$, a new stochastic process indexed by a space of measures $\mathcal{V}$ on a set S, that is determined by a kernel $u(x,y)$, $x,y \in S$, that need not be symmetric. Permanental fields are a generalization of second order Gaussian chaoses. We show that these fields exist whenever $u(x,y)$ is the potential density of a transient Markov process X.

We show that ψ can be realized as the limit of a renormalized sum of continuous additive functionals determined by a loop soup of X. A loop soup is a Poisson point process on the path space of X with an intensity measure μ called the 'loop measure'. (This is done in Section 2.) We obtain a new Dynkin type isomorphism theorem that relates ψ to continuous additive functionals of X and use it to analyze these functionals.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let S be a locally compact metric space with countable base. Let $\mathcal{B}(S)$ denote the Borel σ-algebra, and let $\mathcal{M}(S)$ be the set of finite signed Radon measures on $\mathcal{B}(S)$.

Definition 1.1 *A map ψ from a subset $\mathcal{V} \subseteq \mathcal{M}(S)$ to $\mathcal{F}$ measurable functions on Ω is called an α-permanental field with kernel u if for all $\nu \in \mathcal{V}$, $E\psi(\nu) = 0$ and for all integers $n \geq 2$ and $\nu_1, \ldots, \nu_n \in \mathcal{V}$*

$$E\left(\prod_{j=1}^n \psi(\nu_j)\right) = \sum_{\pi \in \mathcal{P}'} \alpha^{c(\pi)} \int \prod_{j=1}^n u(x_j, x_{\pi(j)}) \prod_{j=1}^n d\nu_j(x_j), \tag{1.1}$$

where $\mathcal{P}'$ is the set of permutations π of [1,n] such that $\pi(j) \neq j$ for any j, and $c(\pi)$ is the number of cycles in the permutation π.

The concept of permanental fields is motivated by [13] and [14, Chapter 9].

The statement in (1.1) makes sense when the kernel u is bounded. However in this case we can accomplish the goals of this paper using permanental processes as we do in [19]. In this paper we are particularly interested in the case in which u is infinite on the diagonal. That is why we define the field using measures on S rather than points in S, and require that $\pi(j) \neq j$ for any j in (1.1), (since we allow $u(x_j, x_j) = \infty$.)

When u is symmetric, positive definite and $\alpha = 1/2$, $\{\psi(\nu)\,,\, \nu \in \mathcal{V}\}$ is given by the Wick square, a particular second order Gaussian chaos defined as

$$:G^2:(\nu) = \lim_{\delta \to 0} \int \left(G_{x,\delta}^2 - E\left(G_{x,\delta}^2\right)\right)\, d\nu(x) \tag{1.2}$$

where $\{G_{x,\delta}, x \in S\}$ is a mean zero Gaussian process with finite covariance $u_\delta(x,y)$, and $\lim_{\delta \to 0} u_\delta(x,y) = u(x,y)$. (See [16] for details.) The results in [16] are simpler to achieve than the results in this paper because we have at our disposal a wealth of information about second order Gaussian chaoses.

The definition of a permanental field in (1.1) is a generalization of the moment formula for permanental processes, introduced in [25]. Let $\theta = \{\theta_x, x \in S\}$ be an α-permanental process with (finite) kernel u, then for any $x_1, \ldots, x_n \in S$

$$E\left(\prod_{j=1}^n \theta_{x_j}\right) = \sum_{\pi \in \mathcal{P}} \alpha^{c(\pi)} \prod_{j=1}^n u(x_j, x_{\pi(j)}), \tag{1.3}$$

where $\mathcal{P}$ is the set of permutations π of [1,n], and $c(\pi)$ is the number of cycles in the permutation π. In this case $\int (\theta_x - E(\theta_x))\, d\nu(x)$ is a permanental field.

Eisenbaum and Kaspi, [5] show that an α-permanental process with kernel u exists whenever u is the potential density of a transient Markov process X in S. (This can also be done using loop soups. See [14, Chapters 2, 4, 5] for a study in the discrete symmetric case.) In [19] we give sufficient conditions for the continuity of α-permanental processes and use this, together with an isomorphism theorem of Eisenbaum and Kaspi, [5] to give sufficient conditions for the joint continuity of the local times of X. In this paper we extend these results to permanental fields and continuous additive functionals.

In order that (1.1) makes sense, we need bounds on multiple integrals of the form

$$\int \prod_{j=1}^n u(x_j, x_{j+1}) \prod_{i=1}^n d\nu_j(x_i), \qquad x_{n+1} = x_1. \tag{1.4}$$

We can accomplish this with proper norms. We say that a norm $\|\cdot\|$ on $\mathcal{M}(S)$ is a **proper norm** with respect to a kernel u if for all $n \geq 2$ and $\nu_1, \ldots, \nu_n$ in $\mathcal{M}(S)$

$$\Big| \int \prod_{j=1}^n u(x_j, x_{j+1}) \prod_{i=1}^n d\nu_j(x_i) \Big| \leq C^n \prod_{j=1}^n \|\nu_j\|, \tag{1.5}$$

for some universal constant $C < \infty$. In Section 6 we present several examples of proper norms which depend upon various hypotheses about the kernel u.

The next step in our program is to show that permanental fields exist. We do this in Section 2 when the kernel $u(x,y)$ is the potential density of a transient Borel right process X in S. (Additional technical conditions are given in Section 2.1.)

We denote by $\mathcal{R}^+(X)$, or $\mathcal{R}^+$ when X is understood, the set of positive bounded Revuz measures ν on S that are associated with X. This is explained in detail in Section 2.1.

Let $\|\cdot\|$ be a proper norm on $\mathcal{M}(S)$ with respect to the kernel u. Set

$$\mathcal{M}^+_{\|\cdot\|} = \{\text{positive } \nu \in \mathcal{M}(S) \,|\, \|\nu\| < \infty\}, \tag{1.6}$$

and

$$\mathcal{R}^+_{\|\cdot\|} = \mathcal{R}^+ \cap \mathcal{M}^+_{\|\cdot\|}. \tag{1.7}$$

Let $\mathcal{M}_{\|\cdot\|}$ and $\mathcal{R}_{\|\cdot\|}$ denote the set of measures of the form $\nu = \nu_1 - \nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}^+_{\|\cdot\|}$ or $\mathcal{R}^+_{\|\cdot\|}$ respectively. We often omit saying that both $\mathcal{R}_{\|\cdot\|}$ and $\|\cdot\|$ depend on the kernel u.

The following theorem is implied by the results in Section 2:

Theorem 1.1 *Let X be a transient Borel right process with state space S and potential density $u(x,y)$, $x, y \in S$, as described in Section 2.1, and let $\|\cdot\|$ be a proper norm with respect to the kernel $u(x,y)$. Then for $\alpha > 0$ we can find an α-permanental field $\{\psi(\nu), \nu \in \mathcal{R}_{\|\cdot\|}\}$ with kernel u.*

We say that $\{\psi(\nu), \nu \in \mathcal{R}_{\|\cdot\|}\}$ is the α-permanental field associated with X.

In Section 3 we study the continuity of permanental fields. Although we need proper norms to prove the existence of permanental fields, the sufficient condition we give for the continuity of permanental fields can be obtained with a weaker assumption on the norm. We say that a norm $\|\cdot\|$ on $\mathcal{M}(S)$ is a weak proper norm with respect to a kernel u, if (1.5) holds for each $\nu \in \mathcal{M}_{\|\cdot\|}$, (that is, when all the ν_j are the same).

Let $\{\psi(\nu), \nu \in \mathcal{V}\}$ be a permanental field with kernel u. Let $\|\cdot\|$ be a weak proper norm with respect to u and suppose that $\mathcal{V} \subseteq \mathcal{M}_{\|\cdot\|}$. We show in Section 3 that

$$\|\psi(\mu) - \psi(\nu)\|_{\Xi} \le C\|\mu - \nu\|, \tag{1.8}$$

where $\|\cdot\|_{\Xi}$ is the norm of the exponential Orlicz space generated by $e^{|x|} - 1$. This inequality enables us to use the well known majorizing measure sufficient condition for the continuity of stochastic processes, (which we state in Section 3), to obtain sufficient conditions for the continuity of permanental fields, $\{\psi(\nu), \nu \in \mathcal{V}\}$ on $(\mathcal{V}, \|\cdot\|)$, where $\|\cdot\|$ denotes the metric $\|\mu - \nu\|$ in (1.8).

Let $B_{\|\cdot\|}(\nu, r)$ denote the closed ball in $(\mathcal{V}, \|\cdot\|)$ with radius r and center ν. For any probability measure σ on $(\mathcal{V}, \|\cdot\|)$ let

$$J_{\mathcal{V},\|\cdot\|,\sigma}(a) = \sup_{\nu \in \mathcal{V}} \int_0^a \log \frac{1}{\sigma(B_{\|\cdot\|}(\nu, r))}\, dr. \tag{1.9}$$

Theorem 1.2 *Let $\{\psi(\nu), \nu \in \mathcal{V}\}$ be an α-permanental field with kernel u and let $\|\cdot\|$ be a weak proper norm for u. Assume that there exists a probability measure σ on $\mathcal{V}$ such that $J_{\mathcal{V},\|\cdot\|,\sigma}(D) < \infty$, where D is the diameter of $\mathcal{V}$ with respect to $\|\cdot\|$ and*

$$\lim_{\delta \to 0} J_{\mathcal{V},\|\cdot\|,\sigma}(\delta) = 0. \tag{1.10}$$

Then ψ is uniformly continuous on $(\mathcal{V}, \|\cdot\|)$ almost surely.

When the kernel u is symmetric, $\{\psi(\nu), \nu \in \mathcal{V}\}$ is a second order Gaussian chaos and it is well known that we can take

$$\|\mu - \nu\| = (E(\psi(\mu) - \psi(\nu))^2)^{1/2}. \tag{1.11}$$

One of of interests in studying permanental fields is to use them to analyze families of continuous additive functionals. We may think of a continuous additive functional of the Markov process $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ as

$$L_t^\nu := \lim_{\epsilon \to 0} \int_S \int_0^t \delta_{y,\epsilon}(X_s)\, ds\, d\nu(y) \tag{1.12}$$

where ν is a positive measure on S and $\delta_{y,\epsilon}$ is an approximate delta function at $y \in S$. More precisely, a family $A = \{A_t\,;\, t \geq 0\}$ of random variables is called a continuous additive functional of X if

(1) $t \mapsto A_t$ is almost surely continuous and nondecreasing, with $A_0 = 0$ and $A_t = A_\zeta$, for all $t \geq \zeta$.

(2) A_t is $\mathcal{F}_t$ measurable.

(3) $A_{t+s} = A_t + A_s \circ \theta_t$ for all $s, t > 0$ a.s.

(Details on the definition of L_t^ν are given in Section 2.)

As in [16] we relate permanental fields and continuous additive functionals by a Dynkin type isomorphism theorem. In Section 4 we obtain such a theorem relating $\{L_\infty^\nu\}$ and the associated permanental field $\{\psi(\nu)\}$. Since the construction of ψ in Section 2 explores many properties of $\{L_\infty^\nu\}$, the further derivation of the isomorphism theorem is relatively straightforward.

In Section 4 we introduce the measure

$$Q_\phi^\rho(1_{\{\zeta > s\}}\, F_s) = \int P^x(F_s\, L_\infty^\phi\, u(X_s, x))\, d\rho(x), \qquad \text{for all } F_s \in b\mathcal{F}_s. \tag{1.13}$$

The next theorem is implied by Theorem 4.1

Theorem 1.3 *Let X be a transient Borel right process with potential densities u, as described in Section 2.1, and let $\|\cdot\|$ be a proper norm for u. Let $\{\psi(\nu), \nu \in \mathcal{R}_{\|\cdot\|}\}$ be the associated α-permanental field with kernel u. Then for any $\phi, \rho \in \mathcal{R}^+_{\|\cdot\|}$ and all measures $\{\nu_j\} \in \mathcal{R}_{\|\cdot\|}$, and all bounded measurable functions F on R^∞,*

$$E\, Q^\rho_\phi \left(F\left(\psi(\nu_i) + L^{\nu_i}_\infty\right)\right) = \frac{1}{\alpha} E\left(\theta^{\rho,\phi}\, F\left(\psi(\nu_i)\right)\right), \tag{1.14}$$

where $\theta^{\rho,\phi}$ is a random variable that has all moments finite.

(Here we use the notation $F(f(x_i)) := F(f(x_1), f(x_2), \ldots)$, and the expectation of the $\{L^{\nu_i}_\infty\}$ are with respect to Q^ρ_ϕ, and of the $\{\psi(\nu_i)\}$ and $\{\theta^{\rho,\phi}\}$ are with respect to E.)

It is easy to show that this isomorphism theorem implies that the continuity of $\{\psi(\nu), \nu \in \mathcal{V}\}$ on $(\mathcal{V}, \|\cdot\|)$, implies the continuity of $\{L^\nu_\infty, \nu \in \mathcal{V}\}$ on $(\mathcal{V}, \|\cdot\|)$). Extending this to the joint continuity of $\{L^\nu_t, (\nu, t) \in \mathcal{V} \times R^+\}$ on $(\mathcal{V} \times R^+, \|\cdot\| \times |\cdot|)$ is considerably more difficult. We do this in Section 5.

Additional hypotheses are required to prove joint continuity of $\{L^\nu_t, (\nu, t) \in \mathcal{V} \times R^+\}$ in the most general setting. However, these are satisfied by a simple sufficient condition when the Markov process is a transient Lévy processes. Let $S = R^d$ and X be a Lévy process killed at the end of an independent exponential time, with characteristic function

$$E e^{i\lambda X_t} = e^{-t\kappa(\lambda)}. \tag{1.15}$$

and potential density $u(x, y) = u(y - x)$. We refer to κ as the characteristic exponent of X.

We assume that

$$\|u\|_2 < \infty \quad \text{and} \quad e^{-\,Re\,\kappa(\xi)} \quad \text{is integrable on } R^d. \tag{1.16}$$

We say that u is radially regular at infinity if

$$\frac{1}{\tau(|\xi|)} \leq |\hat{u}(\xi)| \leq \frac{C}{\tau(|\xi|)} \tag{1.17}$$

where $\tau(|\xi|)$ is regularly varying at infinity. Note that

$$\hat{u}(\xi) = \frac{1}{\kappa(\xi)}. \tag{1.18}$$

For a measure ν on R^d we define the measure ν_h by

$$\nu_h(A) = \nu(A - h). \tag{1.19}$$

Theorem 1.4 *Let $X = \{X(t), t \in R^+\}$ be a Lévy process in R^d that is killed at the end of an independent exponential time, with potential density $u(x, y) = u(y - x)$. Assume that (1.16) holds and $\hat{u}$ is radially regular. Let $\nu \in \mathcal{R}^+(X)$ and $\gamma = |\hat{u}| * |\hat{u}|$. If*

$$\int_1^\infty \frac{(\int_{|\xi| \geq x} |\hat{\nu}(\xi)|^2 \, \gamma(\xi) \, d\xi)^{1/2}}{x} \, dx < \infty, \tag{1.20}$$

then $\{L_t^{\nu_x}, (x, t) \in R^d \times R_+\}$ is continuous P^y almost surely for all $y \in R^d$.

In addition

$$\limsup_{\delta \to 0} \sup_{\substack{|x-y| \leq \delta \\ x,y \in [0,1]^d}} \frac{L_t^{\nu_x} - L_t^{\nu_y}}{\omega(\delta)} \leq C \qquad a.s. \tag{1.21}$$

where

$$\omega(\delta) = \varphi(\delta) \log 1/\delta + \int_0^\delta \frac{\varphi(u)}{u} \, du, \tag{1.22}$$

and

$$\varphi(\delta) = \left(|\delta|^2 \int_{|\xi| \leq 1/|\delta|} |\xi|^2 |\hat{\nu}(\xi)|^2 \gamma(\xi) \, d\xi + \int_{|\xi| \geq 1/|\delta|} |\hat{\nu}(\xi)|^2 \gamma(\xi) \, d\xi \right)^{1/2}. \tag{1.23}$$

Example 1.1

1. If $\tau(|\xi|)$ is regularly varying at infinity with index greater than $d/2$ and less than d and

$$|\hat{\nu}(\xi)| \leq C \frac{\tau(|\xi|)}{|\xi|^d (\log |\xi|)^{3/2+\epsilon}} \qquad \text{as } |\xi| \to \infty \tag{1.24}$$

for some constant $C > 0$ and any $\epsilon > 0$, then $\{L_t^{\nu_x}, (x, t) \in R^d \times R_+\}$ is continuous P^x almost surely.

2. If

$$\tau(|\xi|) = \frac{|\xi|^2}{(\log |\xi|)^a} \qquad \text{for } a \geq 0 \text{ and all } |\xi| \text{ sufficiently large} \tag{1.25}$$

and

$$|\hat{\nu}(\xi)| \leq C \frac{\tau(|\xi|)}{|\xi|^2 (\log |\xi|)^{2+\epsilon}} \qquad \text{as } |\xi| \to \infty \tag{1.26}$$

for some constant $C > 0$ and any $\epsilon > 0$, then $\{L_t^{\nu_x}, (x, t) \in R^2 \times R_+\}$ is continuous P^x almost surely. This extends the result for Brownian motion in R^2 (in which case $a = 0$) that is given in [16, Theorem 1.6].

3. If $\tau(|\xi|)$ is regularly varying at infinity with index $d/2 < \alpha < d$ and

$$|\hat{\nu}(\xi)| \leq \frac{1}{\vartheta(|\xi|)}, \tag{1.27}$$

where $\vartheta(|\xi|)$ is regularly varying at infinity with index β and $\alpha + \beta > d$, then there exists a constant $C > 0$, such that for almost every t,

$$\limsup_{\delta \to 0} \sup_{\substack{|x-y| \leq \delta \\ x,y \in [0,1]^d}} \frac{L_t^{\nu_x} - L_t^{\nu_y}}{\varrho(\delta) \log 1/\delta} \leq C \qquad a.s. \tag{1.28}$$

where

$$\varrho(\delta) \sim C \left(\delta^{-d} \tau(1/\delta) \vartheta(1/\delta)\right)^{-1} \qquad \text{as } \delta \to 0, \tag{1.29}$$

is regularly varying at zero with index $\alpha + \beta - d$.

4. If $d = 2$ and $\tau(|\xi|)$ is as given in (1.25) and

$$|\hat{\nu}(\xi)| \leq \frac{1}{\vartheta(|\xi|)}, \tag{1.30}$$

where $\vartheta(|\xi|)$ is regularly varying at infinity with index $\beta > 0$, then there exists a constant $C > 0$, such that for almost every t

$$\limsup_{\delta \to 0} \sup_{\substack{|x-y| \leq \delta \\ x,y \in [0,1]^d}} \frac{L_t^{\nu_x} - L_t^{\nu_y}}{\varrho(\delta) \log 1/\delta} \leq C \qquad a.s. \tag{1.31}$$

where

$$\varrho(\delta) \sim C \left(\delta^{-2} \tau(1/\delta) \vartheta(1/\delta)\right)^{-1} (\log 1/\delta)^{1/2} \qquad \text{as } \delta \to 0, \tag{1.32}$$

is regularly varying at zero with index β.

Continuous additive functionals of Lévy processes are studied in Section 7.

2 Markov loops and the existence of permanental fields

So far a permanental field is defined as a process with a certain moment structure. In this section we show that a permanental field with kernel u can be realized in terms of continuous additive functionals of a Markov process X with potential density u.

2.1 Continuous additive functionals

Let S a be locally compact set with a countable base. Let $X=(\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient Borel right process with state space S, and jointly measurable transition densities $p_t(x,y)$ with respect to some σ-finite measure m on S. We assume that the potential densities

$$u(x,y)=\int_0^\infty p_t(x,y)\,dt \tag{2.1}$$

are finite off the diagonal, but allow them to be infinite on the diagonal. We also assume that $\sup_x \int_\delta^\infty p_t(x,x)\,dt<\infty$ for each $\delta>0$. We do not require that $p_t(x,y)$ is symmetric.

We assume furthermore that $0<p_t(x,y)<\infty$ for all $0<t<\infty$ and $x,y\in S$, and that there exists another right process $\widehat{X}$ in duality with X, relative to the measure m, so that its transition probabilities $\widehat{P}_t(x,dy)=p_t(x,y)\,m(dy)$. These conditions allow us to use material on bridge measures in [7] in the construction of the loop measure in Section 2.2

Let $\{A_t, t\in R^+\}$ be a positive continuous additive functional of X. The 0-potential of $\{A_t, t\in R^+\}$ is defined to be

$$u_A^0(x)=E^x\left(A_\infty\right). \tag{2.2}$$

If $\{A_t, t\in R^+\}$ and $\{B_t,\, t\in R^+\}$ are two continuous additive functionals of X, with $u_A^0=u_B^0<\infty$, then $\{A_t,\, t\in R^+\}=\{B_t,\, t\in R^+\}$ a.s. (See, e.g., [24, Theorem 36.10].) This can also be seen directly by noting that the properties of a continuous additive functional given in its definition and the Markov property imply that $M_t=A_t-B_t$ is a continuous martingale of bounded variation, and consequently is a constant, [23, IV, (1.2)], which is zero in this case because $M_0=0$.

When $\{A_t, t\in R^+\}$ is a positive continuous additive functional with 0-potential u_A^0 that is the potential of a σ-finite measure ν, that is when,

$$E^x\left(A_\infty\right)=\int u(x,y)\,d\nu(y), \tag{2.3}$$

we write $A_t=L_t^\nu$ and refer to ν as the Revuz measure of A_t.

It follows from [22, V.6] that a σ-finite measure is the Revuz measure of a continuous additive functional of a Markov process X with potential density u if and only if

$$U\nu(x):=\int u(x,y)\,d\nu(y)<\infty \quad \text{for each } x\in S \tag{2.4}$$

and ν does not charge any semi-polar set. We denote by $\mathcal{R}^+(X)$, or $\mathcal{R}^+$ when X is understood, the set of positive bounded Revuz measures. We use $\mathcal{R}$ for the set of measures of the form $\nu = \nu_1 - \nu_2$ with $\nu_1, \nu_2 \in \mathcal{R}^+$, and we write $L_t^\nu = L_t^{\nu_1} - L_t^{\nu_2}$. The comments above show that this is well defined. Throughout this paper we only consider measures in $\mathcal{R}$.

In this paper we consider norms $\|\cdot\|$ on $\mathcal{M}(S)$ that are proper or weak proper norms with respect to the kernel $u(x,y)$. (We often abbreviate this by simply saying that $\|\cdot\|$ a proper or weak proper, norm.) We set

$$\mathcal{M}^+_{\|\cdot\|} := \{\text{positive } \nu \in \mathcal{M}(S) \,|\, \|\nu\| < \infty\}, \tag{2.5}$$

and

$$\mathcal{R}^+_{\|\cdot\|} = \mathcal{R}^+ \cap \mathcal{M}^+_{\|\cdot\|}. \tag{2.6}$$

We use $\mathcal{M}_{\|\cdot\|}, \mathcal{R}_{\|\cdot\|}$ for the set of measures of the form $\nu = \nu_1 - \nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}^+_{\|\cdot\|}$ or $\mathcal{R}^+_{\|\cdot\|}$ respectively. We don't bother to repeat that both $\mathcal{R}_{\|\cdot\|}$ and $\|\cdot\|$ depend on the kernel u.

2.2 The loop measure

It follows from the assumptions in the first two paragraphs of Section 2.1 that, as in [7], for all $0 < t < \infty$ and $x, y \in S$, there exists a finite measure $Q_t^{x,y}$ on $\mathcal{F}_{t^-}$, of total mass $p_t(x,y)$, such that

$$Q_t^{x,y}\left(1_{\{\zeta > s\}}\, F_s\right) = P^x\left(F_s\, p_{t-s}(X_s, y)\right), \tag{2.7}$$

for all $F_s \in \mathcal{F}_s$ with $s < t$. (In this paper we use the letter Q for measures which are not necessarily of mass 1, and reserve the letter P for probability measures.)

We use the canonical representation of X in which Ω is the set of right continuous paths ω in $S_\Delta = S \cup \Delta$ with $\Delta \notin S$, and is such that $\omega(t) = \Delta$ for all $t \geq \zeta = \inf\{t > 0 \,|\omega(t) = \Delta\}$. Set $X_t(\omega) = \omega(t)$. We define a σ-finite measure μ on $(\Omega, \mathcal{F})$ by

$$\mu(F) = \int_0^\infty \frac{1}{t} \int Q_t^{x,x}\left(F \circ k_t\right)\, dm(x)\, dt \tag{2.8}$$

for all $\mathcal{F}$ measurable functions F on Ω. Here k_t is the killing operator defined by $k_t\omega(s) = \omega(s)$ if $s < t$ and $k_t\omega(s) = \Delta$ if $s \geq t$, so that $k_t^{-1}\mathcal{F} \subset \mathcal{F}_{t^-}$. We call μ the loop measure of X because, when X has continuous paths, μ is concentrated on the set of continuous loops with a distinguished starting point (since $Q_t^{x,x}$ is carried by loops starting at x). It can be shown that μ is

invariant under 'loop rotation', and μ is often restricted to the 'loop rotation' invariant sets. We do not pursue these ideas in this paper.

As usual, if F is a function, we often write $\mu(F)$ for $\int F\,d\mu$. (We already used this notation in (2.7)).

We explore some properties of the loop measure μ.

Lemma 2.1 *Let $k \geq 2$, and assume that $\nu_j \in \mathcal{R}_{\|\cdot\|}$ for all $j = 1, \ldots, k$. Then*

$$\mu\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) \tag{2.9}$$

$$= \frac{1}{k} \sum_{\pi \in \mathcal{P}_k} \int u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y_1) \prod_{j=1}^{k} d\nu_{\pi(j)}(y_j)$$

where $\mathcal{P}_k$ denotes the set of permutations of $[1, k]$. Equivalently,

$$\mu\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) = \sum_{\pi \in \mathcal{P}_{k-1}} \int \left(\int u(x, y_1) u(y_1, y_2) \cdots \right. \tag{2.10}$$

$$\left. \cdots u(y_{k-2}, y_{k-1}) u(y_{k-1}, x) \prod_{j=1}^{k-1} d\nu_{\pi(j)}(y_j) \right) d\nu_k(x).$$

When $k = 1$, the formula in (2.9) gives

$$\mu\left(L_\infty^{\nu}\right) = \int u(y, y)\, d\nu(y). \tag{2.11}$$

Obviously, this is infinite when $u(y, y) = \infty$.

Proof We first assume that all the ν_j are positive measures. Note that for all $j = 1, \ldots, k$

$$L_\infty^{\nu_j} \circ k_t = L_t^{\nu_j}. \tag{2.12}$$

Therefore,

$$\begin{aligned} Q_t^{x,x}\left(\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) \circ k_t\right) &= Q_t^{x,x}\left(\prod_{j=1}^{k} L_t^{\nu_j}\right) \\ &= Q_t^{x,x}\left(\prod_{j=1}^{k} \int_0^t dL_{r_j}^{\nu_j}\right) \\ &= \sum_{\pi \in \mathcal{P}_k} Q_t^{x,x}\left(\int_{0 \leq r_1 \leq \cdots \leq r_k \leq t} dL_{r_1}^{\nu_{\pi(1)}} \cdots dL_{r_k}^{\nu_{\pi(k)}}\right). \end{aligned} \tag{2.13}$$

We use the following technical lemma:

Lemma 2.2 *Let $\nu_j \in \mathcal{R}^+$ for all $j = 1, \ldots, k$. Then for all $t \in R^+$*

$$
\begin{aligned}
&Q_t^{x,y}\left(\int_{0 \le r_1 \le \cdots \le r_k \le t} dL_{r_1}^{\nu_1} \cdots dL_{r_k}^{\nu_k}\right) \\
&\quad = \int_{0 \le r_1 \le \cdots \le r_k \le t} \int p_{r_1}(x, y_1) p_{r_2 - r_1}(y_1, y_2) \cdots \\
&\qquad \cdots p_{r_k - r_{k-1}}(y_{k-1}, y_k) p_{t - r_k}(y_k, y) \prod_{j=1}^{k} d\nu_j(y_j)\, dr_j.
\end{aligned}
\tag{2.14}
$$

Proof We prove this by induction on k. The case $k = 1$ follows from [7, Lemma 1]. Assume we have proved (2.14) for all $1 \le j \le k-1$. We write

$$
Q_t^{x,y}\left(\int_{0 \le r_1 \le \cdots \le r_k \le t} dL_{r_1}^{\nu_1} \cdots dL_{r_k}^{\nu_k}\right) = Q_t^{x,y}\left(\int_0^t H_{r_k}\, dL_{r_k}^{\nu_k}\right), \tag{2.15}
$$

where

$$
H_{r_k} = \int_{0 \le r_1 \le \cdots \le r_k} dL_{r_1}^{\nu_1} \cdots dL_{r_{k-1}}^{\nu_{k-1}}. \tag{2.16}
$$

Clearly H_{r_k} is continuous in r_k. It follows from [7, Proposition 3] that

$$
Q_t^{x,y}\left(\int_0^t H_{r_k}\, dL_{r_k}^{\nu_k}\right) = Q_t^{x,y}\left(\int_0^t \frac{Q_{r_k}^{x, X_{r_k}}\left(H_{r_k}\right)}{p_{r_k}(x, X_{r_k})}\, dL_{r_k}^{\nu_k}\right). \tag{2.17}
$$

Using [7, Lemma 1] again we see that

$$
\begin{aligned}
&Q_t^{x,y}\left(\int_0^t \frac{Q_{r_k}^{x, X_{r_k}}\left(H_{r_k}\right)}{p_{r_k}(x, X_{r_k})}\, dL_{r_k}^{\nu_k}\right) \\
&\quad = \int_0^t \left(\int p_{r_k}(x, y_k) p_{t - r_k}(y_k, y) \frac{Q_{r_k}^{x, y_k}\left(H_{r_k}\right)}{p_{r_k}(x, y_k)}\, d\nu_k(y_k)\right) dr_k \\
&\quad = \int_0^t \left(\int p_{t - r_k}(y_k, y) Q_{r_k}^{x, y_k}\left(H_{r_k}\right)\, d\nu_k(y_k)\right) dr_k.
\end{aligned}
\tag{2.18}
$$

Using (2.15) and (2.14) for $k-1$ we see that it holds for all $1 \le j \le k-1$. $\square$

Proof of Lemma 2.1 continued: Combining (2.14) with (2.13) we obtain

$$
\begin{aligned}
&Q_t^{x,x}\left(\prod_{j=1}^k L_\infty^{\nu_j}\circ k_t\right) \qquad (2.19)\\
&\quad=\sum_{\pi\in\mathcal{P}_k}\int_{0\le r_1\le\cdots\le r_k\le t}\int p_{r_1}(x,y_1)p_{r_2-r_1}(y_1,y_2)\cdots\\
&\qquad\qquad\cdots p_{r_k-r_{k-1}}(y_{k-1},y_k)p_{t-r_k}(y_k,x)\prod_{j=1}^k d\nu_{\pi(j)}(y_j)\,dr_j.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int Q_t^{x,x}\left(\prod_{j=1}^k L_\infty^{\nu_j}\circ k_t\right)\,dm(x) \qquad (2.20)\\
&\quad=\sum_{\pi\in\mathcal{P}_k}\int_{0\le r_1\le\cdots\le r_k\le t}\int p_{r_2-r_1}(y_1,y_2)\cdots\\
&\qquad\qquad\cdots p_{r_k-r_{k-1}}(y_{k-1},y_k)p_{r_1+t-r_k}(y_k,y_1)\prod_{j=1}^k d\nu_{\pi(j)}(y_j)\,dr_j,
\end{aligned}
$$

since

$$
\int p_{r_1}(x,y_1)p_{t-r_k}(y_k,x)\,dm(x)=p_{r_1+t-r_k}(y_k,y_1). \qquad (2.21)
$$

It follows from (2.8) and (2.20) that

$$
\begin{aligned}
&\mu\left(\prod_{j=1}^k L_\infty^{\nu_j}\right) \qquad (2.22)\\
&\quad=\sum_{\pi\in\mathcal{P}_k}\int_0^\infty\frac{1}{t}\left(\int_{0\le r_1\le\cdots\le r_k\le t}\int p_{r_2-r_1}(y_1,y_2)\cdots\right.\\
&\qquad\qquad\left.\cdots p_{r_k-r_{k-1}}(y_{k-1},y_k)p_{r_1+t-r_k}(y_k,y_1)\prod_{j=1}^k d\nu_{\pi(j)}(y_j)\,dr_j\right)\,dt.
\end{aligned}
$$

We make the change of variables $(t,r_2\dots,r_k)\to s_1=r_1+t-r_k, s_2=r_2-r_1,\dots,s_k=r_k-r_{k-1}$, and integrate on r_1 to obtain

$$
\begin{aligned}
&\mu\left(\prod_{j=1}^{k} L_{\infty}^{\nu_j}\right) && (2.23)\\
&= \sum_{\pi\in\mathcal{P}_k}\int \frac{1}{s_1+\cdots+s_k}\left(\int p_{s_2}(y_1,y_2)\cdots\right.\\
&\qquad \left.\cdots p_{s_k}(y_{k-1},y_k)p_{s_1}(y_k,y_1)\prod_{j=1}^{k} d\nu_{\pi(j)}(y_j)\right)\left(\int_0^{s_1} 1\,dr_1\right)\prod_{j=1}^{k} ds_j\\
&= \sum_{\pi\in\mathcal{P}_k}\int \frac{s_1}{s_1+\cdots+s_k}\int p_{s_2}(y_1,y_2)\cdots\\
&\qquad \cdots p_{s_k}(y_{k-1},y_k)p_{s_1}(y_k,y_1)\prod_{j=1}^{k} d\nu_{\pi(j)}(y_j)\,ds_j.
\end{aligned}
$$

Set

$$
\begin{aligned}
&f(s_1,s_2,\cdots,s_k) && (2.24)\\
&= \sum_{\pi\in\mathcal{P}_k}\int p_{s_2}(y_1,y_2)\cdots p_{s_k}(y_{k-1},y_k)p_{s_1}(y_k,y_1)\prod_{j=1}^{k} d\nu_{\pi(j)}(y_j),
\end{aligned}
$$

and note that because of the sum over all permutations

$$
f(s_1,s_2,\cdots,s_k)=f(s_2,s_3,\cdots,s_1). \tag{2.25}
$$

Using (2.25) after a simple change of variables we see from (2.23) that

$$
\begin{aligned}
\mu\left(\prod_{j=1}^{k} L_{\infty}^{\nu_j}\right) &= \int \frac{s_1}{s_1+\cdots+s_k} f(s_1,s_2,\cdots,s_k)\prod_{j=1}^{k} ds_j && (2.26)\\
&= \int \frac{s_2}{s_2+s_3+\cdots+s_1} f(s_2,s_3,\cdots,s_1)\prod_{j=1}^{k} ds_j\\
&= \int \frac{s_2}{s_1+s_2+\cdots+s_k} f(s_1,s_2,\cdots,s_k)\prod_{j=1}^{k} ds_j.
\end{aligned}
$$

Similarly we see that for all $1\le j\le k$

$$
\mu\left(\prod_{j=1}^{k} L_{\infty}^{\nu_j}\right) = \int \frac{s_j}{s_1+s_2+\cdots+s_k} f(s_1,s_2,\cdots,s_k)\prod_{j=1}^{k} ds_j. \tag{2.27}
$$

Therefore

$$\mu\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) = \frac{1}{k}\int \frac{s_1+\cdots+s_k}{s_1+\cdots+s_k} f(s_1,s_2,\cdots,s_k)\prod_{j=1}^{k} ds_j \tag{2.28}$$

$$= \frac{1}{k}\int f(s_1,s_2,\cdots,s_k)\prod_{j=1}^{k} ds_j$$

$$= \frac{1}{k}\sum_{\pi\in\mathcal{P}_k}\int\int p_{s_2}(y_1,y_2)\cdots p_{s_k}(y_{k-1},y_k)p_{s_1}(y_k,y_1)$$

$$\prod_{j=1}^{k} d\nu_{\pi(j)}(y_j)\,ds_j$$

$$= \frac{1}{k}\sum_{\pi\in\mathcal{P}_k}\int u(y_1,y_2)\cdots u(y_{k-1},y_k)u(y_k,y_1)\prod_{j=1}^{k} d\nu_{\pi(j)}(y_j).$$

It follows from the hypothesis that $\|\cdot\|$ is a proper norm, that the integrals in (2.28) are finite; consequently the equalities in (2.28) hold for all $\nu\in\mathcal{R}_{\|\cdot\|}$. (I.e. measures that are not necessarily positive.) This is (2.9).

To obtain (2.10) we note that because we are permuting k points on a circle, for $k\geq 2$, we can write (2.9) as

$$\mu\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) = \sum_{\pi\in\mathcal{P}_{k-1}}\int\left(\int u(x,y_1)u(y_1,y_2)\cdots\right. \tag{2.29}$$

$$\left.\cdots u(y_{k-2},y_{k-1})u(y_{k-1},x)\prod_{j=1}^{k-1} d\nu_{\pi(j)}(y_j)\right) d\nu_k(x).$$

□

Remark 2.1 Note that in the course of the proof of Lemma 2.1 we show that (2.9) and (2.10) hold for all measures in $\mathcal{R}^+$.

For use in the next section, note that when $k=1$, (2.20) takes the form

$$\int Q_t^{x,x}\left(L_\infty^{\nu}\circ k_t\right)\,dm(x) = t\int p_t(y,y)\,d\nu(y). \tag{2.30}$$

Using the fact that $1_{\{\zeta>\delta\}}\circ k_t = 1$ if $t>\delta$, and 0 if $t\leq\delta$, we see that

$$\mu(1_{\{\zeta>\delta\}}L_\infty^{\nu}) = \int_\delta^\infty\int p_t(y,y)\,d\nu(y)\,dt \tag{2.31}$$

which is finite by our assumptions that $\sup_x \int_\delta^\infty p_t(x,x)\,dt < \infty$ for each $\delta > 0$ and ν is a finite measure.

2.3 The loop soup

Let $\mathcal{L}_\alpha$ be the Poisson point process on Ω with intensity measure $\alpha\mu$. Note that $\mathcal{L}_\alpha$ is a random variable; each realization of $\mathcal{L}_\alpha$ is a countable subset of Ω. To be more specific, let

$$N(A) := \#\{\mathcal{L}_\alpha \cap A\}, \qquad A \subseteq \Omega. \tag{2.32}$$

Then for any disjoint measurable subsets $A_1, \ldots, A_n$ of Ω, the random variables $N(A_1), \ldots, N(A_n)$, are independent, and $N(A)$ is a Poisson random variable with parameter $\alpha\mu(A)$, i.e.

$$P_{\mathcal{L}_\alpha}\left(N(A) = k\right) = \frac{(\alpha\mu(A))^k}{k!} e^{-\alpha\mu(A)}. \tag{2.33}$$

The Poisson point process $\mathcal{L}_\alpha$ is called the loop soup of the Markov process X. For $\nu \in \mathcal{R}_{\|\cdot\|}$ we define

$$\widetilde{\psi}(\nu) = \lim_{\delta \to 0} \widehat{L}_\delta^\nu, \tag{2.34}$$

where

$$\widehat{L}_\delta^\nu = \left(\sum_{\omega \in \mathcal{L}_\alpha} 1_{\{\zeta(\omega) > \delta\}} L_\infty^\nu(\omega) \right) - \alpha\mu(1_{\{\zeta > \delta\}} L_\infty^\nu). \tag{2.35}$$

As noted following (2.31), $\mu(1_{\{\zeta>\delta\}} L_\infty^\nu)$ is finite for all $\delta > 0$. We show in Theorem 2.1 that the limit (2.34) converges in all L^p, even though each term in (2.35) has an infinite limit as $\delta \to 0$.

The terms loop soup and 'loop soup local time' are used in [11, 12], and [10, Chapter 9]. In [13] they are referred to, less colorfully albeit more descriptively, as Poisson ensembles of Markov loops, and occupation fields of Poisson ensembles of Markov loops.

The next theorem contains Theorem 1.1. It is given for symmetric kernels in [14, Theorem 9]. (In which case, when $\alpha = 1/2$, the permanental process is a second order Gaussian chaos.)

Theorem 2.1 *Let X be a transient Borel right process with state space S and potential density $u(x,y)$, $x, y \in S$, as described in the beginning of this section. Then for $\nu \in \mathcal{R}_{\|\cdot\|}$ the limit (2.34) converges in all L^p and $\{\widetilde{\psi}(\nu), \nu \in \mathcal{R}_{\|\cdot\|}\}$ is an α-permanental field with kernel $u(x,y)$.*

Proof By the master formula for Poisson processes, [9, (3.6)],

$$E_{\mathcal{L}_\alpha}\left(e^{\sum_{j=1}^n z_j \widehat{L}^{\nu_j}_{\delta_j}}\right) \tag{2.36}$$
$$= \exp\left(\alpha\left(\int_\Omega\left(e^{\sum_{j=1}^n z_j 1_{\{\zeta>\delta_j\}}L^{\nu_j}_\infty}-\sum_{j=1}^n z_j 1_{\{\zeta>\delta_j\}}L^{\nu_j}_\infty-1\right)\,d\mu(\omega)\right)\right).$$

Differentiating each side of (2.36) with respect to $z_1,\ldots,z_n$ and then setting $z_1,\ldots,z_n$ equal to zero, we see that

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^n \widehat{L}^{\nu_j}_{\delta_j}\right)=\sum_{\cup_i B_i=[1,n],\,|B_i|\geq 2}\ \prod_i \alpha\ \mu\left(\prod_{j\in B_i}1_{\{\zeta>\delta_j\}}L^{\nu_j}_\infty\right), \tag{2.37}$$

where the sum is over all partitions $B_1,\ldots,B_n$ of $[1,n]$ with all $|B_i|\geq 2$.

The right hand side of (2.37) can be written as a sum of terms involving only positive measures, to which the monotone convergence theorem can be applied. Using (2.10) we then see that the right hand side has a limit as the $\delta_j\to 0$ and this limit is the same as the right-hand side of (1.1). Applying this with $\prod_{j=1}^n \widehat{L}^{\nu_j}_{\delta_j}$ replaced by $\left(\widehat{L}^{\nu}_{\delta}-\widehat{L}^{\nu}_{\delta'}\right)^n$, for arbitrary integer n, shows that the limit (2.34) exists in all L^p. $\square$

Remark 2.2 *If we let α vary, we get a field-valued process with independent stationary increments. This property is inherited from the analogous property of the loop soup.*

3 Continuity of Permanental fields

In this section we give sufficient conditions for the continuity of second order permanental fields that extend well known results for second order Gaussian chaoses.

We use the following well known sufficient condition for continuity of stochastic processes with a metric in an exponential Orlicz space, see e.g., [18, Section 3]. Let $\Xi(x)=\exp(x)-1$ and $L^\Xi(\Omega,\mathcal{F},P)$ denote the set of random variables $\xi:\Omega\to R^1$ such that $E(\Xi(|\xi|/c))<\infty$ for some $c>0$. $L^\Xi(\Omega,\mathcal{F},P)$ is a Banach space with norm given by

$$\|\xi\|_\Xi=\inf\left\{c>0: E(\Xi(|\xi|/c))\leq 1\right\}. \tag{3.1}$$

Let (T,τ) be a metric or pseudo-metric space. Let $B_\tau(t,u)$ denote the closed ball in (T,τ) with radius u and center t. For any probability measure σ on (T,τ) we define

$$J_{T,\tau,\sigma}(a) = \sup_{t\in T}\int_0^a \log\frac{1}{\sigma(B_\tau(t,u))}\,du. \tag{3.2}$$

Theorem 3.1 *Let $X=\{X(t): t\in T\}$ be a stochastic process such that $X(t,\omega): T\times\Omega\mapsto[-\infty,\infty]$ is $\mathcal{A}\times\mathcal{F}$ measurable for some σ-algebra $\mathcal{A}$ on T. Suppose $X(t)\in L^\rho(\Omega,\mathcal{F},P)$ and there exists a metric τ on T such that*

$$\|X(s)-X(t)\|_\Xi \le \tau(s,t). \tag{3.3}$$

(Note that the balls $B_\tau(s,u)$ are $\mathcal{A}$ measurable).

Suppose that (T,τ) has finite diameter D, and that there exists a probability measure σ on $(T,\mathcal{A})$ such that

$$J_{T,\tau,\sigma}(D)<\infty. \tag{3.4}$$

Then there exists a version $X'=\{X'(t), t\in T\}$ of X such that

$$E\sup_{t\in T}X'(t)\le C\,J_{T,\tau,\sigma}(D), \tag{3.5}$$

for some $C<\infty$. Furthermore for all $0<\delta\le D$,

$$\sup_{\substack{s,t\in T\\ \tau(s,t)\le\delta}} |X'(s,\omega)-X'(t,\omega)|\le 2Z(\omega)\,J_{T,\tau,\sigma}(\delta), \tag{3.6}$$

almost surely, where

$$Z(\omega):=\inf\left\{\alpha>0: \int_T \phi(\alpha^{-1}|X(t,\omega)|)\,\sigma(dt)\le 1\right\} \tag{3.7}$$

and $\|Z\|_\Xi\le K$, where K is a constant.

In particular, if

$$\lim_{\delta\to 0}J_{T,\tau,\sigma}(\delta)=0, \tag{3.8}$$

X' is uniformly continuous on (T,τ) almost surely.

The next lemma shows the significance of weak proper norms for continuity results for permanental field.

Lemma 3.1 *Let $\{\psi(\nu), \nu \in \mathcal{V}\}$ be an α-permanental field with kernel u and let $\|\cdot\|$ be a weak proper norm with respect to u, then for some $C_\alpha < \infty$, depending only on α,*

$$\|\psi(\nu)\|_\Xi \leq C_\alpha \|\nu\|, \qquad \forall \nu \in \mathcal{V} \tag{3.9}$$

and

$$\|\psi(\nu) - \psi(\nu')\|_\Xi \leq C_\alpha \|\nu - \nu'\|, \qquad \forall \nu, \nu' \in \mathcal{V}, \tag{3.10}$$

where $\|\cdot\|_\Xi$ is the norm of the exponential Orlicz space generated by $e^{|x|} - 1$.

Proof of Lemma 3.1 We note the following immediate consequence of Definition 1.1

$$E\left((\psi(\nu) - \psi(\nu'))^n\right) = E(\psi^n(\nu - \nu')). \tag{3.11}$$

Therefore to estimate $E\left((\psi(\nu) - \psi(\nu'))^n\right)$ it suffices to consider $E(\psi^n(\nu - \nu'))$, or simply $E(\psi^n(\nu))$, where ν is not required to be a positive measure.

To prove this lemma we must estimate $E(|\psi(\nu)|^n)$. When n is even we can use Definition 1.1. Consider a term on the right-hand side of (1.1)

$$\alpha^{c(\pi)} \int \prod_{j=1}^n u(x_j, x_{\pi(j)}) \prod_{j=1}^n d\nu(x_j) \tag{3.12}$$

for a fixed permutation $\pi \in \mathcal{P}'$. Since $c(\pi)$ is the number of cycles in π, we see that $c(\pi) \leq n/2$. Therefore $\alpha^{c(\pi)} \leq (\alpha \vee 1)^{n/2}$.

Suppose π has p cycles of lengths $l_1, \ldots, l_p$. Label the elements in cycle q, $q_1, \ldots, q_{l_q}$. Consider the terms on the right- hand side of (3.12) corresponding to the qth cycle

$$\int \prod_{j=1}^{l_q} u(x_{q_j}, x_{q_{j+1}}) \prod_{j=1}^{l_q} d\nu(x_{q_j}), \tag{3.13}$$

where $l_q + 1 = 1$. Since $\|\cdot\|$ is a weak proper norm we have

$$\int \prod_{j=1}^{l_q} u(x_{q_j}, x_{q_{j+1}}) \prod_{j=1}^{l_q} d\nu(x_{q_j}) \leq C^{l_q} \|\nu\|^{l_q} \tag{3.14}$$

Therefore (3.12)

$$\leq (\alpha \vee 1)^{n/2} C^n \|\nu\|^n. \tag{3.15}$$

It now follows from Definition 1.1 that for n even

$$E|\psi(\nu)|^n \leq (n-1)!\, (\alpha \vee 1)^{n/2} C^n \|\nu\|^n. \tag{3.16}$$

When n is odd by Hölders inequality

$$\begin{aligned} E|\psi(\nu)|^n &\leq (E|\psi(\nu)|^{n+1})^{n/(n+1)} \\ &\leq n!\,(\alpha\vee 1)^{n/2}C^n\|\nu\|^n. \end{aligned} \tag{3.17}$$

Consequently,

$$E\left(\frac{|\psi(\nu)|}{2C(\alpha\vee 1)^{1/2}\|\nu\|}\right)^n \leq \frac{n!}{2^n}. \tag{3.18}$$

This gives (3.9) and (3.10). □

Let $\mathcal{V}$ be a linear space of measures on S and u a kernel on $S\times S$. Suppose that $\|\cdot\|$ is a weak proper norm for $\mathcal{V}$ with respect to u and $\mathcal{V}\subseteq\mathcal{R}_{\|\cdot\|}$. Then $\|\nu-\nu'\|$ is a metric on $\mathcal{V}$ and is a candidate for the metric τ in (3.3). In this situation we write $J_{T,\tau,\sigma}$ in (3.2) as $J_{\mathcal{V},\|\cdot\|,\sigma}$.

Proof of Theorem 1.2 This is an immediate consequence of Lemma 3.1 and Theorem 3.1.

Other results on the continuity of permanental fields are given in [20].

Remark 3.1 All the other results in Theorem 3.1, that are not stated in Theorem 1.2, hold for ψ and $\|\cdot\|$.

4 The Isomorphism Theorem

In this section we obtain an isomorphism theorem that relates permanental fields and continuous additive functionals. To begin we consider properties of several measures on the probability space of X. Recall that u denotes the 0-potential density of X

Let $Q^{x,y}$ denote the σ-finite measure defined by

$$Q^{x,y}(1_{\{\zeta>s\}}\ F_s)=P^x(F_s\ u(X_s,y))\qquad\text{for all } F_s\in b\mathcal{F}^0_s, \tag{4.1}$$

where $\mathcal{F}^0_s$ is the σ-algebra generated by $\{X_r,\,0\leq r\leq s\}$.

Lemma 4.1 *For all x,y*

$$Q^{x,y}(F)=\int_0^\infty Q^{x,y}_t\,(F\circ k_t)\ dt,\qquad F\in b\mathcal{F}^0. \tag{4.2}$$

Proof To obtain (4.2) it suffices to prove it for F of the form $\{1_{\{\zeta>s\}}\,F_s\}$ for all $F_s\in b\mathcal{F}^0_s$. Since $1_{\{\zeta>s\}}\circ k_t=1_{\{s>t\}}1_{\{\zeta>s\}}$,

$$\begin{aligned}\int_0^\infty Q^{x,y}_t\left((1_{\{\zeta>s\}}\,F_s)\circ k_t\right)\,dt &= \int_s^\infty Q^{x,y}_t\left(1_{\{\zeta>s\}}\,F_s\right)\,dt \\ &= \int_s^\infty P^x\left(F_s\,p_{t-s}(X_s,y)\right)\,dt \\ &= P^x(F_s\;u(X_s,y))=Q^{x,y}\left(1_{\{\zeta>s\}}\,F_s\right),\end{aligned}\tag{4.3}$$

where the second and third equalities follow from (2.7) and interchanging the order of integration and the final equation by (4.1). □

We have the following formula for the moments of $\{L^\nu_\infty,\nu\in\mathcal{R}^+\}$ under $Q^{x,y}$.

Lemma 4.2 *For all $\nu_j\in\mathcal{R}^+$, $j=1,\ldots,k$,*

$$\begin{aligned}Q^{x,y}\left(\prod_{j=1}^k L^{\nu_j}_\infty\right) &= \sum_{\pi\in\mathcal{P}_k}\int u(x,y_1)u(y_1,y_2)\cdots \\ &\qquad\cdots u(y_{k-1},y_k)u(y_k,y)\prod_{j=1}^k d\nu_{\pi(j)}(y_j),\end{aligned}\tag{4.4}$$

where the $\mathcal{P}_k$ denotes the set of permutations of $[1,k]$.

Proof By (4.2), taking into account the observation in (**??**), we have

$$Q^{x,y}\left(\prod_{j=1}^k L^{\nu_j}_\infty\right) = \int_0^\infty Q^{x,y}_t\left(\prod_{j=1}^k L^{\nu_j}_\infty\circ k_t\right)\,dt.\tag{4.5}$$

Following the argument in (2.13) and then using Lemma 2.2 we see that

$$\begin{aligned}Q^{x,y}_t\left(\left(\prod_{j=1}^k L^{\nu_j}_\infty\right)\circ k_t\right) &= \sum_{\pi\in\mathcal{P}_k}Q^{x,y}_t\left(\int_{0\le r_1\le\cdots\le r_k\le t}dL^{\nu_{\pi(1)}}_{r_1}\cdots dL^{\nu_{\pi(k)}}_{r_k}\right) \\ &= \sum_{\pi\in\mathcal{P}_k}\int_{0\le r_1\le\cdots\le r_k\le t}\int p_{r_1}(x,y_1)p_{r_2-r_1}(y_1,y_2)\cdots \\ &\qquad p_{r_k-r_{k-1}}(y_{k-1},y_k)p_{t-r_k}(y_k,y)\prod_{j=1}^k d\nu_{\pi(j)}(y_j)\,dr_j.\end{aligned}$$

Therefore

$$Q^{x,y}\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) = \sum_{\pi\in\mathcal{P}_k}\int_{0\le r_1\le\cdots\le r_k\le t<\infty}\int p_{r_1}(x,y_1)p_{r_2-r_1}(y_1,y_2)\cdots$$
$$p_{r_k-r_{k-1}}(y_{k-1},y_k)p_{t-r_k}(y_k,y)\prod_{j=1}^{k} d\nu_{\pi(j)}(y_j)\,dr_j\,dt,$$

which gives (4.4). □

Let $\|\cdot\|$ be a proper norm. It follows from (4.4) and (2.10) that for any ρ and $\nu_1,\ldots,\nu_k\in\mathcal{R}_{\|\cdot\|}$

$$\int Q^{x,x}\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right)\,d\rho(x) = \mu\left(L_\infty^{\rho}\prod_{j=1}^{k} L_\infty^{\nu_j}\right). \tag{4.6}$$

For any $\phi,\rho\in\mathcal{R}^+_{\|\cdot\|}$, set

$$Q_\phi^\rho(A) = \int Q^{x,x}(L_\infty^\phi 1_{\{A\}})\,d\rho(x). \tag{4.7}$$

Note that by (4.6) we have $Q_\phi^\rho(\Omega)=\mu(L_\infty^\rho L_\infty^\phi)$, so that Q_ϕ^ρ is a finite measure. Using (4.6) again, we see that

$$Q_\phi^\rho\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) = \mu\left(L_\infty^\rho L_\infty^\phi\prod_{j=1}^{k} L_\infty^{\nu_j}\right) \tag{4.8}$$

for all $\nu_j\in\mathcal{R}_{\|\cdot\|}$.

By (2.10) and (4.8) and the fact that $\|\cdot\|$ is a proper norm we see that

$$|Q_\phi^\rho\left((L_\infty^\nu)^n\right)| = |\mu\left(L_\infty^\rho L_\infty^\phi\,(L_\infty^\nu)^n\right)| \le n!C^n\|\phi\|\|\rho\|\|\nu\|^n. \tag{4.9}$$

Therefore L_∞^ν is exponentially integrable with respect to the finite measures $Q_\phi^\rho(\cdot)$ and $\mu\left(L_\infty^\rho L_\infty^\phi(\cdot)\right)$, so that the finite dimensional distributions of $\{L_\infty^\nu,\ \nu\in\mathcal{R}_{\|\cdot\|}\}$ under these measures are determined by their moments. Consequently, by (4.8), for all bounded measurable functions F on R^k,

$$Q_\phi^\rho\left(F\left(L_\infty^{\nu_1},\ldots,L_\infty^{\nu_k}\right)\right) = \mu\left(L_\infty^\rho L_\infty^\phi\ F\left(L_\infty^{\nu_1},\ldots,L_\infty^{\nu_k}\right)\right). \tag{4.10}$$

We now obtain a Dynkin type isomorphism theorem that relates permanental fields with kernel u to continuous additive functionals of a Markov process with potential density u. We can do this very efficiently by employing a special case of the Palm formula for Poisson processes $\mathcal{L}$ with intensity measure ξ on a measurable space $\mathcal{S}$, see [1, Lemma 2.3], which states that for any positive function f on $\mathcal{S}$ and any measurable functional G of $\mathcal{L}$

$$E_{\mathcal{L}}\left(\left(\sum_{\omega\in\mathcal{L}} f(\omega)\right) G(\mathcal{L})\right) = \int E_{\mathcal{L}}\left(G(\omega'\cup\mathcal{L})\right) f(\omega')\, d\xi(\omega'). \tag{4.11}$$

For $\phi,\rho\in\mathcal{R}^{+}_{\|\cdot\|}$ we define

$$\theta^{\rho,\phi} = \sum_{\omega\in\mathcal{L}_\alpha} L^{\rho}_{\infty}(\omega) L^{\phi}_{\infty}(\omega). \tag{4.12}$$

Obviously, $\theta^{\rho,\phi}\geq 0$.

Theorem 4.1 (Isomorphism Theorem I) *Let X be a transient Borel right process with potential density u as described in Section 2.1. Let $\|\cdot\|$ be a proper norm for u and let $\{\widetilde{\psi}(\nu),\nu\in\mathcal{R}_{\|\cdot\|}\}$ be as described in (2.34). (By Theorem 2.1, $\{\widetilde{\psi}(\nu),\nu\in\mathcal{R}_{\|\cdot\|}\}$ is an α-permanental field with kernel u.) Let $\{L^{\nu}_{\infty},\nu\in\mathcal{R}_{\|\cdot\|}\}$ be as described in the paragraph containing (2.3). Then for any $\phi,\rho\in\mathcal{R}_{\|\cdot\|}$ and all measures $\nu_j\in\mathcal{R}_{\|\cdot\|}$, $j=1,2,\ldots,$ and all bounded measurable functions F on R^{∞},*

$$E_{\mathcal{L}_\alpha} Q^{\rho}_{\phi}\left(F\left(\widetilde{\psi}(\nu_i)+L^{\nu_i}_{\infty}\right)\right) = \frac{1}{\alpha} E_{\mathcal{L}_\alpha}\left(\theta^{\rho,\phi} F\left(\widetilde{\psi}(\nu_i)\right)\right), \tag{4.13}$$

and $\theta^{\rho,\phi}$, given in (4.12), has all its moments finite. (Here we use the notation $F(f(x_i)):=F(f(x_1),f(x_2),\ldots)$.)

Since (4.13) depends only on the distribution of the α-permanental field with kernel u, this theorem implies Theorem 1.3.

Proof We apply the Palm formula with intensity measure $\alpha\mu$,

$$f(\omega) = L^{\rho}_{\infty}(\omega) L^{\phi}_{\infty}(\omega) \tag{4.14}$$

and

$$G(\mathcal{L}_\alpha) = F\left(\widehat{L}^{\nu_j}_{\delta}\right). \tag{4.15}$$

To begin let F be a bounded continuous function on R^n. Note that

$$\sum_{\omega\in\mathcal{L}_\alpha} f(\omega) = \theta^{\rho,\phi}. \tag{4.16}$$

Note also that since ω' and $\mathcal{L}_\alpha$ are disjoint a.s.

$$\begin{aligned}\widehat{L}^{\nu_j}_\delta(\omega'\cup\mathcal{L}_\alpha) &= \left(\sum_{\omega\in\omega'\cup\mathcal{L}_\alpha} 1_{\{\zeta(\omega)>\delta\}}L^{\nu_j}_\infty(\omega)\right)-\alpha\mu(1_{\{\zeta>\delta\}}L^{\nu_j}_\infty) \\ &= 1_{\{\zeta(\omega')>\delta\}}L^{\nu_j}_\infty(\omega')+\widehat{L}^{\nu_j}_\delta(\mathcal{L}_\alpha), \end{aligned} \tag{4.17}$$

so that

$$G(\omega'\cup\mathcal{L}_\alpha)=F\left(\widehat{L}^{\nu_j}_\delta(\mathcal{L}_\alpha)+1_{\{\zeta(\omega')>\delta\}}L^{\nu_j}_\infty(\omega')\right). \tag{4.18}$$

It follows from (4.11) that

$$\begin{aligned}& E_{\mathcal{L}_\alpha}\left(\theta^{\rho,\phi}F\left(\widehat{L}^{\nu_j}_\delta\right)\right) \qquad (4.19) \\ &\quad =\alpha\int E_{\mathcal{L}_\alpha}\left(L^\rho_\infty(\omega')L^\phi_\infty(\omega')\,F\left(\widehat{L}^{\nu_j}_\delta(\mathcal{L}_\alpha)+1_{\{\zeta(\omega')>\delta\}}L^{\nu_j}_\infty(\omega')\right)\right)\,d\mu(\omega').\end{aligned}$$

We interchange the integrals on the right-hand side of (4.19) and use (4.10) and then take the limit as $\delta\to 0$, to get (4.13) for bounded continuous functions F on R^n. The extension to general bounded measurable functions F on R^∞_+ is routine.

To see that $\theta^{\rho,\phi}$ has all moments finite, we use the master formula for Poisson processes in the form

$$E_{\mathcal{L}_\alpha}\left(e^{z\theta^{\rho,\phi}}\right)=\exp\left(\alpha\left(\int_\Omega\left(e^{zL^\rho_\infty L^\theta_\infty}-1\right)\,d\mu(\omega)\right)\right) \tag{4.20}$$

with $z<0$. Differentiating each side of (4.20) n times with respect to z and then taking $z\uparrow 0$ we see that

$$E_{\mathcal{L}_\alpha}\left((\theta^{\rho,\phi})^n\right)=\sum_{\cup_i B_i=[1,n]}\prod_i \alpha\,\mu\left((L^\rho_\infty L^\theta_\infty)^{|B_i|}\right), \tag{4.21}$$

where the sum is over all partitions $B_1,\ldots,B_n$ of $[1,n]$. This is finite for $\phi,\rho\in\mathcal{R}^+_{\|\cdot\|}$. □

Isomorphism Theorem I shows that the continuity of the permanental field implies the continuity (in the measures) of the continuous additive functionals.

Corollary 4.1 *In the notation and under the hypotheses of Theorem 4.1, let* $D\subseteq\mathcal{R}_{\|\cdot\|}$ *and suppose there exists a metric* d *on* D *such that*

$$\lim_{\delta\to 0}E\left(\sup_{\substack{d(\nu,\nu')\le\delta\\ \nu,\nu'\in D}}\left|\psi(\nu)-\psi(\nu')\right|^2\right)=0, \tag{4.22}$$

then

$$\lim_{\delta\to 0} Q^{\rho}_{\phi}\left(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}\left|L^{\nu}_{\infty}-L^{\nu'}_{\infty}\right|\right)=0. \tag{4.23}$$

Proof It follows from (4.13) that

$$\begin{aligned} Q^{\rho}_{\phi}&\left(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}|L^{\nu}_{\infty}-L^{\nu'}_{\infty}|\right) \qquad (4.24)\\ &\leq E\Bigg(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}\left|\psi(\nu)-\psi(\nu')\right|\Bigg)\; Q^{\rho}_{\phi}(1)\\ &\quad+\frac{1}{\alpha}\Bigg(E\Bigg(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}\left|\psi(\nu)-\psi(\nu')\right|^2\Bigg)E(\theta^{\rho,\phi})^2\Bigg)^{1/2}. \end{aligned}$$

Using this it is easy to see that (4.22) implies (4.23). □

For applications of the Isomorphism Theorem in Section 5 we sometimes need to consider measures ρ and ϕ in (4.13) that are not necessarily in $\mathcal{R}_{\|\cdot\|}$. To deal with this we introduce two additional norms on $\mathcal{M}(S)$:

$$\|\nu\|_{u^2,\infty}:=|\nu|(S)\vee\sup_x\int\left(u^2(x,y)+u^2(y,x)\right)\,d|\nu|(y), \tag{4.25}$$

where $|\nu|$ is the total variation of the measure ν, and

$$\|\nu\|_0:=|\nu|(S)\vee\sup_x\int u(x,y)\,d|\nu|(y). \tag{4.26}$$

Lemma 4.3 *Let $A\cup B$ be a partition of $[1,n]$, $n\geq 2$, with $B\neq\emptyset$. Then*

$$\begin{aligned} \Big|\int u(y_1,y_2)\cdots u(y_{n-1},y_n)u(y_n,y_1)\prod_{j=1}^n d\nu_j(y_j)\Big| \qquad (4.27)\\ \leq\prod_{i\in A}\|\nu_i\|_0\prod_{j\in B}\|\nu_j\|_{u^2,\infty}. \end{aligned}$$

Let $\phi\in\mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$ and $\rho\in\mathcal{R}^+_{\|\cdot\|_0}$. In addition let and $\nu_j\in\mathcal{R}_{\|\cdot\|}$, $j=1,\ldots,k$, for some proper norm $\|\cdot\|$. Then there exists a constant $C=C(\phi,\rho,\|\cdot\|)<\infty$, such that

$$\left|\mu\left(L^{\rho}_{\infty}L^{\phi}_{\infty}\prod_{j=1}^k L^{\nu_j}_{\infty}\right)\right|\leq k!C^k\,\|\rho\|_0\;\|\phi\|_{u^2,\infty}\prod_{j=1}^k\|\nu_j\|. \tag{4.28}$$

Proof Without loss of generality, we assume that $1 \in B$. Then using the Cauchy-Schwarz inequality in y_1 we have

$$\begin{aligned}
&\Big| \int u(y_1,y_2)\cdots u(y_{k-1},y_k)u(y_k,y_1) \prod_{j=1}^{k} d\nu_j(y_j) \Big| \qquad (4.29)\\
&\quad \le \int \left(\int u^2(y_1,y_2)\, d|\nu_1|(y_1) \right)^{1/2} \left(\int u^2(y_k,y_1)\, d|\nu_1|(y_1) \right)^{1/2} \\
&\qquad\qquad u(y_2,y_3)\cdots u(y_{k-1},y_k) \prod_{j=2}^{k} d|\nu_j|(y_j). \\
&\quad \le \|\nu_1\|_{u^2,\infty} \int u(y_2,y_3)\cdots u(y_{k-1},y_k) \prod_{j=2}^{k} d|\nu_j|(y_j).
\end{aligned}$$

We bound successively the integrals with respect to $d|\nu_j|(y_j)$ for $j = k, k-1, \ldots, 3$ to obtain

$$\begin{aligned}
&\int u(y_2,y_3)\cdots u(y_{k-1},y_k) \prod_{j=2}^{k} d|\nu_j|(y_j). \qquad (4.30)\\
&\quad \le \left(\sup_x \int u(x,y)\, d|\nu_k|(y) \right) \int |u(y_2,y_3)\cdots u(y_{k-2},y_{k-1}) \prod_{j=2}^{k-1} d|\nu_j|(y_j) \\
&\quad \le \prod_{j=3}^{k} \left(\sup_x \int u(x,y)\, d|\nu_j|(y) \right) \int 1 \; d|\nu_2|(y_2) \le \prod_{j=2}^{k} \|\nu_j\|_0.
\end{aligned}$$

Using (4.29) and (4.30) we see that

$$\Big| \int u(y_1,y_2)\cdots u(y_{n-1},y_n)u(y_n,y_1) \prod_{j=1}^{n} d\nu_j(y_j) \Big| \le \|\nu_1\|_{u^2,\infty} \prod_{j=2}^{n} \|\nu_j\|_0. \quad (4.31)$$

We now note that by the Cauchy-Schwarz inequality $\|\nu\|_0^{1/2} \le \|\nu\|_{u^2,\infty}^{1/2}$. Using this and (4.31) and recognizing that the choice of indices in (4.31) is arbitrary we get (4.27).

To obtain (4.28) we use the Cauchy-Schwarz inequality to write

$$\Big|\mu\Big(L^\rho_\infty L^\phi_\infty \prod_{j=1}^{k} L^{\nu_j}_\infty \Big)\Big| \le \Big\{ \mu\left((L^\rho_\infty L^\phi_\infty)^2 \right) \Big\}^{1/2} \Big\{ \mu\Big(\Big(\prod_{j=1}^{k} L^{\nu_j}_\infty \Big)^2 \Big) \Big\}^{1/2}. \quad (4.32)$$

We use (4.27) with $|A| = |B| = 2$ to bound the first term and (2.10) and (1.5), and the fact that $((2k-1)!)^{1/2} \leq C^k k!$ to bound the second term and get (4.28). $\square$

Using Lemma 4.3 we can modify the hypotheses of Theorem 4.1 to obtain a second isomorphism theorem.

Theorem 4.2 (Isomorphism Theorem II) *All the results of Theorem 4.1 hold for* $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$ *and* $\rho \in \mathcal{R}^+_{\|\cdot\|_0}$.

Proof Given the proof of Theorem 4.1, to prove this theorem it suffices to show that (4.10) holds when $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$ and $\rho \in \mathcal{R}^+_{\|\cdot\|_0}$. To do this we first show that the argument from (4.6)–(4.10) holds under this change of hypothesis.

Set

$$Q^\rho_\phi(A) = \int Q^{x,x}(L^\phi_\infty 1_{\{A\}})\, d\rho(x). \tag{4.33}$$

By Remark 2.1, (2.10) holds for measures in $\mathcal{R}^+$. In particular, for $\rho, \nu_j \in \mathcal{R}^+$

$$\int Q^{x,x}\left(\prod_{j=1}^k L^{\nu_j}_\infty\right)\, d\rho(x) = \mu\left(L^\rho_\infty \prod_{j=1}^k L^{\nu_j}_\infty\right). \tag{4.34}$$

Therefore, $Q^\rho_\phi(\Omega) = \mu(L^\rho_\infty L^\phi_\infty)$ so that by Lemma 4.3 (4.27), we see that Q^ρ_ϕ is a finite measure. Using (4.34) and (2.10) we see that

$$Q^\rho_\phi\left(\prod_{j=1}^k L^{\nu_j}_\infty\right) = \mu\left(L^\rho_\infty L^\phi_\infty \prod_{j=1}^k L^{\nu_j}_\infty\right) \tag{4.35}$$

for all $\nu_j \in \mathcal{R}^+$.

We now use Lemma 4.3 (4.28), to see that (4.35) holds for $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$, $\rho \in \mathcal{R}^+_{\|\cdot\|_0}$ and $\{\nu_j\} \in \mathcal{R}_{\|\cdot\|}$. Therefore, using Lemma 4.3 we see that that for any $\nu \in \mathcal{R}_{\|\cdot\|}$

$$|Q^\rho_\phi\left((L^\nu_\infty)^n\right)| = \mu\left(L^\rho_\infty L^\phi_\infty \left(L^\nu_\infty\right)^n\right) \leq n! C^n \,\|\rho\|_0\, \|\phi\|_{u^2,\infty} \|\nu\|^n, \tag{4.36}$$

which shows that all $\{L^{\nu_j}_\infty\}$ are exponentially integrable with respect to the finite measures Q^ρ_ϕ and $\mu\left(L^\rho_\infty L^\phi_\infty\, \cdot\right)$. Since (4.35) holds for $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$, $\rho \in$

$\mathcal{R}^+_{\|\cdot\|_0}$ and $\{\nu_j\} \in \mathcal{R}_{\|\cdot\|}$, this shows that for all bounded measurable functions F on R^k

$$Q^\rho_\phi \left(F \left(L^{\nu_1}_\infty, \ldots, L^{\nu_k}_\infty \right) \right) = \mu \left(L^\rho_\infty L^\phi_\infty \; F \left(L^{\nu_1}_\infty, \ldots, L^{\nu_k}_\infty \right) \right). \tag{4.37}$$

holds when $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$, $\rho \in \mathcal{R}^+_{\|\cdot\|_0}$ and $\{\nu_j\} \in \mathcal{R}_{\|\cdot\|}$. With this modification the proof of Theorem 4.1 proves this theorem. $\square$

Remark 4.1 It is easy to see that Corollary 4.1 also holds under the hypotheses of Theorem 4.2.

5 Joint continuity of continuous additive functionals

In this section we obtain sufficient conditions for continuity of the stochastic process

$$L = \left\{ L^\nu_t, (t,\nu) \in R^+ \times \mathcal{V} \right\}, \tag{5.1}$$

for some family of measures $\mathcal{V} \subseteq \mathcal{R}^+$, endowed with a topology induced by an appropriate proper norm..

By definition L^ν_t is continuous in t. However, proving the joint continuity of (5.1), P^x almost surely, is difficult. We break the proof into a series of lemmas and theorems. We assume that $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$ and $\rho \in \mathcal{R}^+_{\|\cdot\|_0}$, which, as noted above, implies that Q^ρ_ϕ, (defined in (4.33)), is a finite measure.

Let $h(x,y)$ be a bounded measurable function on S which is excessive in x, and such that

$$0 < h(x,y) \leq u(x,y), \qquad x, y \in S. \tag{5.2}$$

For example, we can take $h(x,y) = 1 \wedge u(x,y)$, or more generally $h(x,y) = f(x) \wedge u(x,y)$ for any bounded strictly positive excessive function f. In the proof of Theorem 1.4 we take $h(x,y) = u_1(x,y) = \int_1^\infty p_t(x,y)\, dt$.

Set $h_y(z) = h(z,y)$. We let Q^{x,h_y} denote the (finite) measure defined by

$$Q^{x,h_y}(1_{\{\zeta > s\}} \; F_s) = P^x(F_s \; h(X_s, y)) \qquad \text{for all } F_s \in b\mathcal{F}^0_s, \tag{5.3}$$

where $\mathcal{F}^0_s$ is the σ-algebra generated by $\{X_r,\, 0 \leq r \leq s\}$. In this notation, we can write the σ-finite measure $Q^{x,y}$ in (4.1) as Q^{x,u_y}.

Set

$$Q^{x,h_x}_\phi(A) = Q^{x,h_x}(L^\phi_\infty 1_{\{A\}}) \tag{5.4}$$

and

$$Q_\phi^{\rho,h}(A)=\int Q^{x,h_x}(L_\infty^\phi 1_{\{A\}})\,d\rho(x)=\int Q_\phi^{x,h_x}(A)\,d\rho(x). \tag{5.5}$$

Note that it follows from (5.2) and (5.3) that

$$Q_\phi^{\rho,h}(A)\leq Q_\phi^{\rho}(A) \tag{5.6}$$

for all $A\in\mathcal{F}^0$.

Lemma 5.1 *Let $X=(\Omega,X_t,P^x)$ be a Borel right process in S with strictly positive potential densities $u(x,y)$, and let $\mathcal{V}\subseteq\mathcal{R}^+_{\|\cdot\|}$, where $\|\cdot\|$ is proper for u. Let $\mathcal{O}$ be a topology for $\mathcal{V}$ under which $\mathcal{V}$ is a separable locally compact metric space with metric d. Assume that there exist measures $\rho\in\mathcal{R}^+_{\|\cdot\|_0}$, and $\phi\in\mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$ for which*

(i)

$$\int u(y,z)h_x(z)\,d\nu(z)\quad\textit{and}\quad\int u(y,z)\int u(z,w)h_x(w)\,d\phi(w)\,d\nu(z) \tag{5.7}$$

are continuous in $\nu\in\mathcal{V}$, uniformly in $y,x\in S$, and

(ii) *for any countable set $D\subseteq\mathcal{V}$, with compact closure*

$$\lim_{\delta\to 0}Q_\phi^\rho\left(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}\left|L_\infty^\nu-L_\infty^{\nu'}\right|\right)=0, \tag{5.8}$$

where $\{L_t^\nu,\nu\in D\}$ are continuous additive functionals of X as defined in Section 2.1.

Then for any $\epsilon>0$, there exists a $\delta>0$, such that

$$Q_\phi^{\rho,h}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}L_t^\nu-L_t^{\nu'}\geq 2\epsilon\right)\leq\epsilon. \tag{5.9}$$

Proof As in [16] we use martingale techniques to go from (5.8) to (5.9). However, the present situation is considerably more complicated.

By working locally it suffices to consider $\mathcal{V}$ compact. For fixed y, let

$$P^{x/h_y}(\cdot)=\frac{Q^{x,h_y}(\cdot)}{h_y(x)},\qquad x\in S. \tag{5.10}$$

$(\Omega, X_t, P^{x/h_y})$ is a Borel right process in S, called the h_y-transform of (Ω, X_t, P^x), [24, Section 62].

To begin we fix $x \in S$. Set

$$Z = \frac{L^{\phi}_{\infty}}{E^{x/h_x}(L^{\phi}_{\infty})} \quad \text{and} \quad Z_s = E^{x/h_x}(Z \mid \mathcal{F}^0_s), \tag{5.11}$$

and define the probability measure

$$P^{x/h_x}_{\phi}(A) := E^{x/h_x}(1_{\{A\}} Z) = \frac{Q^{x,h_x}_{\phi}(A)}{Q^{x,h_x}\left(L^{\phi}_{\infty}\right)}. \tag{5.12}$$

By [17, Lemma 3.9.1] we can assume that the continuous additive functionals L^{ν}_t are $\mathcal{F}^0_t$ measurable. Consider the P^{x/h_x}_{ϕ} martingale

$$A^{\nu}_s = E^{x/h_x}_{\phi}(L^{\nu}_{\infty} \mid \mathcal{F}^0_s) = \frac{E^{x/h_x}(L^{\nu}_{\infty} Z \mid \mathcal{F}^0_s)}{Z_s}. \tag{5.13}$$

The last equality is well known and easy to check. Using the additivity property

$$L^{\nu}_{\infty} = L^{\nu}_s + L^{\nu}_{\infty} \circ \tau_s, \tag{5.14}$$

where τ_s denotes the shift operator on Ω, we see that

$$A^{\nu}_s = L^{\nu}_s + \frac{E^{x/h_x}(L^{\nu}_{\infty} \circ \tau_s \; Z \mid \mathcal{F}^0_s)}{Z_s} := L^{\nu}_s + H^{\nu}_s. \tag{5.15}$$

Let D be a countable dense subset of $\mathcal{V}$. By (5.15), for any finite subset $F \subset D$,

$$\begin{aligned}
P^{x/h_x}_{\phi} & \left(\sup_{t \geq 0} \sup_{\substack{d(\nu,\nu') \leq \delta \\ \nu,\nu' \in F}} L^{\nu}_t - L^{\nu'}_t \geq 3\epsilon \right) \qquad (5.16) \\
&\leq P^{x/h_x}_{\phi} \left(\sup_{t \geq 0} \sup_{\substack{d(\nu,\nu') \leq \delta \\ \nu,\nu' \in F}} A^{\nu}_t - A^{\nu'}_t \geq \epsilon \right) \\
&\quad + P^{x/h_x}_{\phi} \left(\sup_{t \geq 0} \sup_{\substack{d(\nu,\nu') \leq \delta \\ \nu,\nu' \in F}} H^{\nu}_t - H^{\nu'}_t \geq 2\epsilon \right) \\
&:= I_{1,x} + I_{2,x}.
\end{aligned}$$

Using (5.14), but this time for L^{ϕ}_{∞}, and using the Markov property, we see that

$$\begin{aligned} H^{\nu}_t &= \frac{E^{x/h_x}(L^{\nu}_{\infty}\circ\tau_t\ L^{\phi}_{\infty}\mid\mathcal{F}^0_t)}{E^{x/h_x}(L^{\phi}_{\infty}\mid\mathcal{F}^0_t)} \qquad (5.17)\\ &= \frac{L^{\phi}_t\ E^{x/h_x}(L^{\nu}_{\infty}\circ\tau_t\mid\mathcal{F}^0_t)+E^{x/h_x}((L^{\nu}_{\infty}L^{\phi}_{\infty})\circ\tau_t\mid\mathcal{F}^0_t)}{E^{x/h_x}(L^{\phi}_{\infty}\mid\mathcal{F}^0_t)}\\ &= \frac{L^{\phi}_t\ E^{X_t/h_x}(L^{\nu}_{\infty})+E^{X_t/h_x}(L^{\nu}_{\infty}L^{\phi}_{\infty})}{E^{x/h_x}(L^{\phi}_{\infty}\mid\mathcal{F}^0_t)}. \end{aligned}$$

Here and throughout we are using the convention that $f(X_t)=1_{\{t>\zeta\}}f(X_t)$ for any function f on S. Proceeding the same way with the denominator we obtain

$$H^{\nu}_t=\frac{L^{\phi}_t\ E^{X_t/h_x}(L^{\nu}_{\infty})+E^{X_t/h_x}(L^{\nu}_{\infty}L^{\phi}_{\infty})}{L^{\phi}_t+E^{X_t/h_x}(L^{\phi}_{\infty})}. \qquad (5.18)$$

Using [16, (2.25) and (2.22)] where $u\beta(\,\cdot\,)=h_x(\,\cdot\,)$, we have

$$E^{y/h_x}(L^{\nu}_{\infty})=\frac{\int u(y,z)h_x(z)\,d\nu(z)}{h_x(y)} \qquad (5.19)$$

and

$$\begin{aligned} E^{y/h_x}(L^{\nu}_{\infty}L^{\phi}_{\infty}) &= \frac{\int u(y,w)\left(\int u(w,z)h_x(z)\,d\nu(z)\right)\,d\phi(w)}{h_x(y)} \qquad (5.20)\\ &\quad +\frac{\int u(y,w)\left(\int u(w,z)h_x(z)\,d\phi(z)\right)\,d\nu(w)}{h_x(y)}. \end{aligned}$$

By assumption (i) these are finite, and since they are excessive in y it follows that H^{ν}_t is right continuous in t. Hence it follows from (5.13) that A^{ν}_t, $t\geq 0$ is also right continuous. Therefore

$$\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in\overline{F}}} A^{\nu}_t-A^{\nu'}_t=\sup_{\substack{d(\nu,z)\leq\delta\\ \nu,\nu'\in\overline{F}}}|A^{\nu}_t-A^{\nu'}_t| \qquad (5.21)$$

is a right continuous, non–negative submartingale and therefore, using (5.13), we see that

$$\begin{aligned} I_{1,x} &= P^{x/h_x}_{\phi}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in\overline{F}}} A^{\nu}_t-A^{\nu'}_t\geq\epsilon\right) \qquad (5.22)\\ &\leq \frac{1}{\epsilon}E^{x/h_x}_{\phi}\left(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in\overline{D}}}|L^{\nu}_{\infty}-L^{\nu'}_{\infty}|\right). \end{aligned}$$

Using (5.5) and then (5.12)

$$Q_{\phi}^{\rho,h}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in F}} L_t^{\nu}-L_t^{\nu'}\geq 3\epsilon\right) \tag{5.23}$$
$$=\int Q_{\phi}^{x,h_x}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in F}} L_t^{\nu}-L_t^{\nu'}\geq 3\epsilon\right)d\rho(x)$$
$$=\int P_{\phi}^{x/h_x}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in F}} L_t^{\nu}-L_t^{\nu'}\geq 3\epsilon\right)\ Q^{x,h_x}\left(L_{\infty}^{\phi}\right)d\rho(x),$$

so that by (5.16) and (5.22)

$$Q_{\phi}^{\rho,h}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in F}} L_t^{\nu}-L_t^{\nu'}\geq 3\epsilon\right) \tag{5.24}$$
$$\leq\frac{1}{\epsilon}\int E_{\phi}^{x/h_x}\left(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}|L_{\infty}^{\nu}-L_{\infty}^{\nu'}|\right)\ Q^{x,h_x}\left(L_{\infty}^{\phi}\right)d\rho(x)$$
$$+\int P_{\phi}^{x/h_x}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in F}} H_t^{\nu}-H_t^{\nu'}\geq 2\epsilon\right)\ Q^{x,h_x}\left(L_{\infty}^{\phi}\right)d\rho(x)$$
$$:=I_\delta+II_\delta. \tag{5.25}$$

Using (5.12) and then (5.5), we see that

$$I_\delta=\frac{1}{\epsilon}Q_{\phi}^{\rho,h}\left(\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}}|L_{\infty}^{\nu}-L_{\infty}^{\nu'}|\right) \tag{5.26}$$

It follows from (5.6) and assumption ii that for any $\epsilon'>0$, we can choose a $\delta>0$, for which (5.26) is less that ϵ'.

We show below that

$$\lim_{\delta\to 0}P_{\phi}^{x/h_x}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}} H_t^{\nu}-H_t^{\nu'}\geq 2\epsilon\right)Q^{x,h_x}(L_{\infty}^{\phi})=0, \tag{5.27}$$

uniformly in x. Considering (5.24), the proof is completed by taking $F \uparrow D$.

To prove (5.27) we use (5.17) to write

$$
\begin{aligned}
&P_\phi^{x/h_x}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}} H_t^\nu - H_t^{\nu'}\geq 2\epsilon\right) \qquad (5.28)\\
&\quad\leq P_\phi^{x/h_x}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}} \frac{h_x(X_t)\,L_t^\phi\,\left(E^{X_t/h_x}(L_\infty^\nu)-E^{X_t/h_x}(L_\infty^{\nu'})\right)}{h_x(X_t)E^{x/h_x}(L_\infty^\phi\mid\mathcal{F}_t^0)}\geq\epsilon\right)\\
&\quad+P_\phi^{x/h_x}\left(\sup_{t\geq 0}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}} \frac{h_x(X_t)\left(E^{X_t/h_x}(L_\infty^\nu L_\infty^\phi)-E^{X_t/h_x}(L_\infty^{\nu'}L_\infty^\phi)\right)}{h_x(X_t)E^{x/h_x}(L_\infty^\phi\mid\mathcal{F}_t^0)}\geq\epsilon\right).
\end{aligned}
$$

Let

$$
\gamma_x(\delta) = \sup_{y\in S}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}} h_x(y)|E^{y/h_x}(L_\infty^\nu)-E^{y/h_x}(L_\infty^{\nu'})|
$$

and

$$
\bar\gamma_x(\delta) = \sup_{y\in S}\sup_{\substack{d(\nu,\nu')\leq\delta\\ \nu,\nu'\in D}} h_x(y)|E^{y/h_x}(L_\infty^\nu L_\infty^\phi)-E^{y/h_x}(L_\infty^{\nu'}L_\infty^\phi)|. \qquad (5.29)
$$

Then the first line of (5.28) is less than or equal to

$$
\begin{aligned}
&P_\phi^{x/h_x}\left(\sup_{t\geq 0}\frac{L_t^\phi\gamma_x(\delta)}{h_x(X_t)\;E^{x/h_x}(L_\infty^\phi\mid\mathcal{F}_t^0)}\geq\epsilon\right) \qquad (5.30)\\
&\quad+P_\phi^{x/h_x}\left(\sup_{t\geq 0}\frac{\bar\gamma_x(\delta)}{h_x(X_t)\;E^{x/h_x}(L_\infty^\phi\mid\mathcal{F}_t^0)}\geq\epsilon\right).
\end{aligned}
$$

It follows from (5.19), (5.20) and assumption (i) that

$$
\lim_{\delta\to 0}\gamma_x(\delta)=0 \qquad \text{and} \qquad \lim_{\delta\to 0}\bar\gamma_x(\delta)=0 \qquad (5.31)
$$

uniformly in $x\in S$. Consequently, bounding L_t^ϕ by $E^x(L_\infty^\phi\mid\mathcal{F}_t^0)$ in the first line of (5.30), we see that (5.27) follows from the next lemma. □

Lemma 5.2 *Let M_t be a non-negative right continuous P^x martingale. Then*

$$\frac{M_t}{h_x(X_t)E^{x/h_x}(L^\phi_\infty \mid \mathcal{F}^0_t)}, \qquad t \geq 0 \tag{5.32}$$

is a right continuous non-negative supermartingale with respect to P^{x/h_x}_ϕ, and

$$P^{x/h_x}_\phi \left(\sup_{t\geq 0} \frac{M_t}{h_x(X_t)E^{x/h_x}(L^\phi_\infty \mid \mathcal{F}^0_t)} \geq \epsilon \right) \leq \frac{1}{\epsilon} \frac{P^x(M_0)}{Q^{x,h_x}(L^\phi_\infty)}. \tag{5.33}$$

Proof For any $t > s \geq 0$ and any $F_s \in \mathcal{F}^0_s$ we have

$$\begin{aligned} J &:= P^{x/h_x}_\phi \left(F_s \frac{M_t}{h_x(X_t)E^{x/h_x}(L^\phi_\infty \mid \mathcal{F}^0_t)} \right) \qquad (5.34)\\ &= \frac{1}{E^{x/h_x}(L^\phi_\infty)} P^{x/h_x} \left(L^\phi_\infty F_s \frac{M_t}{h_x(X_t)E^{x/h_x}(L^\phi_\infty \mid \mathcal{F}^0_t)} \right) \\ &= \frac{1}{E^{x/h_x}(L^\phi_\infty)} P^{x/h_x} \left(F_s \frac{M_t}{h_x(X_t)} \right) \end{aligned}$$

Note that for all functions f on S, $f(\Delta) = 0$. Therefore, using (5.3) and (5.10)

$$P^{x/h_x} \left(F_s \frac{M_t}{h_x(X_t)} \right) = P^{x/h_x} \left(1_{\{\zeta>t\}} F_s \frac{M_t}{h_x(X_t)} \right) = \frac{P^x \left(1_{\{\zeta>t\}} F_s M_t \right)}{h_x(x)}. \tag{5.35}$$

Consequently

$$\begin{aligned} J &= \frac{1}{h_x(x)E^{x/h_x}(L^\phi_\infty)} P^x \left(1_{\{\zeta>t\}} F_s M_t \right) \\ &\leq \frac{1}{h_x(x)E^{x/h_x}(L^\phi_\infty)} P^x \left(1_{\{\zeta>s\}} F_s M_t \right) \\ &= \frac{1}{h_x(x)E^{x/h_x}(L^\phi_\infty)} P^x \left(1_{\{\zeta>s\}} F_s M_s \right). \end{aligned}$$

Considering (5.35) and (5.34) with t replaced by s we see that the last line above is equal to

$$P^{x/h_x}_\phi \left(F_s \frac{M_s}{h_x(X_s)E^{x/h_x}(L^\phi_\infty \mid \mathcal{F}^0_s)} \right). \tag{5.36}$$

This shows that (5.32) is a non-negative supermartingale with respect to P_ϕ^{x/h_x}. That it is right continuous follows from

$$E^{x/h_x}(L_\infty^\phi \mid \mathcal{F}_t^0) = L_t^\phi + E^{X_t/h_x}(L_\infty^\phi) \tag{5.37}$$

and the sentence following (5.20). This and the fact that $h_x(x)E^{x/h_x}(L_\infty^\phi) = Q^{x,h_x}(L_\infty^\phi)$ gives (5.33). □

We can now give our most general result about the joint continuity of the continuous additive functionals.

Theorem 5.1 *Assume that conditions (i) and (ii) in Lemma 5.1 are satisfied for some $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$ with support $\phi = S$, and some $\rho \in \mathcal{R}^+_{\|\cdot\|_0}$ of the form $\rho(dx) = f(x)\,m(dx)$ with $f > 0$. Then there exists a version of $\left\{L_t^\nu,(t,\nu) \in R^1_+ \times \mathcal{V}\right\}$ that is continuous on $(0,\zeta) \times \mathcal{V}$, P^x almost surely for all $x \in S$, and is continuous on $[0,\zeta) \times \mathcal{V}$, P^x almost surely for $m(dx)$ a.e. $x \in S$. (Continuity on $\mathcal{V}$ is with respect to the metric d introduced in the statement of Lemma 5.1.)*

Proof By (ii) of Lemma 5.1 $\{L_t^\nu,(t,\nu) \in R^+ \times D\}$ is uniformly continuous on $\mathcal{V}$, uniformly in $t \in R^+$ with respect to $Q_\phi^{\rho,h}$. The first step in this proof is to show that $\{L_t^\nu,(t,\nu) \in R^+ \times D\}$ is locally uniformly continuous almost surely with respect to $Q_\phi^{\rho,h}$. This can be proved by mimicking the proof in [15, Theorem 6.1] that (6.10) implies (6.12). (This theorem is given for a different family of continuous additive functionals with different conditions on the potential density of the associated Markov process, nevertheless it is not difficult to see that a straightforward adaptation of the proof works in the case we are considering.)

Let

$$\tilde\Omega_1 = \left\{\omega \,|\, L_t^\nu(\omega) \text{ is locally uniformly continuous on } R^+ \times D\right\}. \tag{5.38}$$

We have that

$$Q_\phi^{\rho,h}(\tilde\Omega_1^c) = \int Q_\phi^{x,h_x}(\tilde\Omega_1^c)\,d\rho(x) = 0. \tag{5.39}$$

Using the fact that $L_\infty^\phi > 0$ and $\rho(dx) = f(x)\,m(dx)$ with $f > 0$, we see from (5.39) that

$$Q^{x,h_x}(\tilde\Omega_1^c) = 0, \text{ for } m(dx) \text{ a.e. } x \in S. \tag{5.40}$$

Set

$$\tilde\Omega_2 = \{\omega \,|\, L_t^\nu(\omega) \text{ is locally uniformly continuous on } [0,\zeta) \times D\}. \tag{5.41}$$

We see from (5.40) and (5.3) that

$$P^x(\tilde{\Omega}_2^c)=0,\ \text{for } m(dx) \text{ a.e. } x\in S. \tag{5.42}$$

Because the Markov process has positive transition densities, we see that for any $x\in S$ and $\epsilon>0$

$$P^x(\tilde{\Omega}_2^c\circ\theta_\epsilon)=E^x\left(P^{X_\epsilon}(\tilde{\Omega}_2^c)\right)=\int p_\epsilon(x,y)P^y(\tilde{\Omega}_2^c)\,dm(y)=0. \tag{5.43}$$

Consequently, for

$$\tilde{\Omega}_3:=\{\omega\,|\,L_t^\nu(\omega) \text{ is locally uniformly continuous on } (0,\zeta)\times D\}, \tag{5.44}$$

we have

$$P^x(\tilde{\Omega}_3^c)=0,\ \text{for all } x\in S. \tag{5.45}$$

For $\omega\in\tilde{\Omega}_3^c$ we set $\tilde{L}_t^\nu(\omega)\equiv 0$. For $\omega\in\tilde{\Omega}_3$ we define $\{\tilde{L}_t^\nu(\omega),(t,\nu)\in(0,\zeta)\times\mathcal{V}\}$ as the continuous extension of $\{L_t^\nu(\omega),(t,\nu)\in(0,\zeta)\times D\}$, and then set

$$\tilde{L}_0^\nu(\omega)=\liminf_{\substack{s\downarrow 0\\ s \text{ rational}}}\tilde{L}_s^\nu(\omega) \tag{5.46}$$

and

$$\tilde{L}_t^\nu(\omega)=\liminf_{\substack{s\uparrow\zeta(\omega)\\ s \text{ rational}}}\tilde{L}_s^\nu(\omega),\quad \text{for all } t\geq\zeta. \tag{5.47}$$

Since $L_t^\nu(\omega)$ is increasing in t for $\nu\in D$, the same is true for $\{\tilde{L}_t^\nu(\omega),(t,\nu)\in(0,\zeta)\times\mathcal{V}\}$. Therefore the lim infs in (5.46) and (5.47) are actually limits. Since we can assume that the L_t^ν are perfect continuous additive functionals for all $\nu\in D$, we immediately see that the same is true for $\tilde{L}_t^\nu$ for each $\nu\in\mathcal{V}$, except that one problem remains. We need $\tilde{L}_0^\nu=0$, but it is not clear from (5.46) that this is the case.

We show that $\tilde{L}_t^\nu$ is a version of L_t^ν, which implies that $\tilde{L}_0^\nu=0$. Pick some ν' not in D and set $D'=D\cup\{\nu'\}$. Then by the argument above, but with D replaced by D', we get that $L_t^\nu(\omega)$ is locally uniformly continuous on $(0,\zeta)\times D'$, almost surely. Thus $L_t^{\nu'}=\widetilde{L}_t^{\nu'}$ on $(0,\zeta)$ a.s., which is enough to show that $\{\widetilde{L}_t^{\nu'},t\geq 0\}$ is a version of $\{L_t^{\nu'},t\geq 0\}$.

Thus we see that there exists a version of $\left\{L_t^\nu,(t,\nu)\in R_+^1\times\mathcal{V}\right\}$ that is continuous on $(0,\zeta)\times\mathcal{V}$, P^x almost surely for all $x\in S$. To see that this version is continuous on $[0,\zeta)\times\mathcal{V}$, P^x almost surely for $m(dx)$ a.e. $x\in S$, it suffices to note that for each $\omega\in\Omega_2$, $\tilde{L}_t^\nu(\omega)$ is continuous on $[0,\zeta)\times\mathcal{V}$, and then use (5.42). □

We now take $S = R^n$. Let T_a denote the bijection on the space of measures defined by the translation $T_a(\nu) = \nu_a$; see (1.19). We say that a set $\mathcal{V}$ of measures on R^n is translation invariant if it is invariant under T_a for each $a \in R^n$ and say that a topology $\mathcal{O}$ on such a set $\mathcal{V}$ is homogeneous if T_a is an isomorphism for each $a \in R^n$.

Theorem 5.2 *Let X be an exponentially killed Lévy process in R^n and $\mathcal{V} \subseteq \mathcal{R}^+_{\|\cdot\|}$ be a translation invariant set of measures on R^n. Assume*

(i) *that there is a homogeneous topology $\mathcal{O}$ for $\mathcal{V}$ under which $\mathcal{V}$ is a separable locally compact metric space with metric d, and*

(ii) *that conditions (i) and (ii) in Lemma 5.1 are satisfied for some $\phi \in \mathcal{R}^+_{\|\cdot\|_{u^2,\infty}}$ with support $\phi = S$, and some $\rho \in \mathcal{R}^+_{\|\cdot\|_0}$ of the form $\rho(dx) = f(x)\, m(dx)$ with $f > 0$.*

Then there exists a version of $\left\{L^\nu_t, (t,\nu) \in R^1_+ \times \mathcal{V}\right\}$ that is continuous P^x almost surely for all $x \in S$.

Proof Using the fact that X is an exponentially killed process it follows easily from the proof of Theorem 5.1 and [16, p. 1149] that we can replace ζ by ∞ in the conclusions of Theorem 5.1. Hence there exists a version of $\left\{L^\nu_t, (t,\nu) \in R^1_+ \times \mathcal{V}\right\}$ that is continuous P^x almost surely for a.e. $x \in S$. By translation invariance this holds for all $x \in S$. $\square$

By Corollary 4.1 we can replace condition (ii) of Lemma 5.1 by (4.22). This is used to obtain the next corollary that allows us to replace condition (ii) in Lemma 5.1 by a more concrete condition that follows from Theorems 3.1 and 1.2.

Corollary 5.1 *Let $X = (\Omega, X_t, P^x)$ be a Borel right process in S with strictly positive 0-potential densities $u(x,y)$, and let $\mathcal{V}$ be a separable locally compact subset of $\mathcal{R}^+_{\|\cdot\|}$. Assume that there exists a probability measure σ on $\mathcal{V}$ such that $J_{\mathcal{V},\|\cdot\|,\sigma}(D) < \infty$, where D is the diameter of $\mathcal{V}$ with respect to $\|\cdot\|$, and*

$$\lim_{\delta \to 0} J_{\mathcal{V},\|\cdot\|,\sigma}(\delta) = 0. \tag{5.48}$$

Then condition (ii) of Lemma 5.1 holds.

6 Proper norms

Given a family of measures $\mathcal{V}$ on a space S and a kernel $\{u(x,y), x, y \in S\}$, a proper norm $\|\cdot\|$ for $\mathcal{V}$ with respect to u is a norm for which (1.5) holds. In this section we give many proper norms, depending on various hypotheses for the kernels u. One of the difficulties in obtaining proper norms is that we do not require that the measures $\nu \in \mathcal{V}$ are positive. (Even if we are only interested in a set $\widetilde{\mathcal{V}}$ of positive measures, as in Theorem 1.4, we would still need to consider $\|\nu - \nu'\|$ for $\nu, \nu' \in \widetilde{\mathcal{V}}$, as, for example, in (3.10), and in general $\nu - \nu'$ is not a positive measure.)

We present the results in two theorems. In the first we consider that the kernel $u(x,y)$ is the potential density of a Markov process X, with state space S and transition density $p_t(x,y)$ so that $u(x,y) = \int_0^\infty p_t(x,y)\,dt$.

Theorem 6.1 *For a kernel $u(x,y), x, y \in S$ that is the potential density of a Markov process X with transition densities $p_t(x,y)$, satisfying the given additional conditions, the following norms are proper norms on $\mathcal{M}(S)$ with respect to u:*

1. *$S = R^d$, X is an exponentially killed Lévy process with $u(x,y) = u(y-x)$ and Fourier transform $\hat{u}$ and*

$$\gamma(x) := (|\hat{u}| * |\hat{u}|)(x). \tag{6.1}$$

$$\begin{aligned} \|\nu\|_{\gamma,2} &:= \left(\int |\hat{\nu}(x)|^2 \, \gamma(x)\, dx \right)^{1/2} \\ &= C' \left(\int\!\!\int \left(\int e^{i(x-y)q} |\hat{u}(q)|\, dq \right)^2 d\nu(x) d\nu(y) \right)^{1/2}. \end{aligned} \tag{6.2}$$

1a. *Same as in 1. and in addition $\kappa(x)$, the characteristic exponent of X, (i.e., $Ee^{i\lambda X_t} = e^{-t\kappa(\lambda)}$), satisfies the sectorial condition*

$$|Im\ \kappa(x)| \le C\ Re\ \kappa(x), \qquad \forall x \in R^m \tag{6.3}$$

for some constant $C < 1$. Then

$$\|\nu\|_{[2]} := \|\psi(\nu)\|_2 = \left(\int\!\!\int u(x,y) u(y,x)\, d\nu(x)\, d\nu(y) \right)^{1/2}, \tag{6.4}$$

where, as usual, $\|\cdot\|_2$ is the L^2 norm.

2. *For general S*

$$\|\nu\|_w := \left(\int\int \left(\int w(x,y)w(y,z)\, d\nu(y) \right)^2 dm(x)\, dm(z) \right)^{1/2}, \quad (6.5)$$

where m is the reference measure for X and

$$w(x,y) = \int_0^\infty \frac{p_s(x,y)}{\sqrt{\pi s}}\, ds. \quad (6.6)$$

3. *For general S*

$$\|\nu\|_\Phi := \left(\int\int \Phi(x,y)\, d\nu(x) d\nu(y) \right)^{1/2}, \quad (6.7)$$

where

$$\begin{aligned} \Phi(x,y) &= \Theta_l(x,y)\Theta_r(x,y), \quad (6.8) \\ \Theta_l(x,y) &= \int_0^\infty \int p_{s/2}(x,u) p_{s/2}(y,u)\, dm(u)\, ds, \\ \Theta_r(x,y) &= \int_0^\infty \int p_{s/2}(u,x) p_{s/2}(u,y)\, dm(u)\, ds \end{aligned}$$

and m is the reference measure for X.

Proof The results in 1, 1a, and 3 change of number are proved [20, Section 3]. The results in 1, and 3 are consequences of of a general inequality, [20, Lemma 3.2]. Nevertheless, considering the importance of $\|\nu\|_{\gamma,2}$ in Section 7 we give a simple self-contained proof for 1.

We write u as the inverse Fourier transform of it's Fourier transform to get

$$\begin{aligned} &\Big| \int \prod_{j=1}^n u(z_{j+1} - z_j) \prod_{j=1}^n d\nu_j(z_j) \Big| \quad (6.9) \\ &= \frac{1}{(2\pi)^{nd}} \Big| \int\int \prod_{j=1}^n e^{-i(z_{j+1}-z_j)\cdot x_j} \hat{u}(x_j)\, dx_j d\nu_j(z_j) \Big| \\ &= \frac{1}{(2\pi)^{nd}} \Big| \int \left(\prod_{j=1}^n \int e^{i(x_j - x_{j-1})\cdot z_j}\, d\nu_j(z_j) \hat{u}(x_j)\, dx_j \right) \Big| \end{aligned}$$

where $x_0 = x_n$. We take the Fourier transforms of the $\{\nu_j\}$ to see that (6.9)

$$\begin{aligned} &\le \frac{1}{(2\pi)^{nd}} \int\!\!\int \left(\prod_{j=1}^{n} |\hat{\nu}_j(x_j - x_{j-1}) \hat{u}(x_j)| \, dx_j \right) && (6.10) \\ &= \frac{1}{(2\pi)^{nd}} \int\!\!\int \left(\prod_{j=1}^{n} |\hat{\nu}_j(x_j - x_{j-1})| \; |\hat{u}(x_j)|^{1/2} \; |\hat{u}(x_{j-1})|^{1/2} \, dx_j \right). \end{aligned}$$

We now note that for any sequence of real or complex valued functions $\{w_j\}_{j=1}^n$, and with $x_0 = x_n$, by repeated use of the Cauchy-Schwarz Inequality,

$$\int \cdots \int \left(\prod_{j=1}^{n} |w_j(x_j, x_{j-1})| \, dx_j \right) \le \prod_{j=1}^{n} \left(\int \int |w_j(x,y)|^2 \, dx \, dy \right)^{1/2}. \quad (6.11)$$

We apply (6.11) to the last line of (6.10) with

$$|w_j(x_j, x_{j-1})| = \frac{1}{(2\pi)^d} |\hat{\nu}_j(x_j - x_{j-1})| |\hat{u}(x_j)|^{1/2} |\hat{u}(x_{j-1})|^{1/2} \quad (6.12)$$

to get that (6.9)

$$\begin{aligned} &= \prod_{j=1}^{n} \left(\frac{1}{(2\pi)^d} \int \int |\hat{\nu}_j(x-y)|^2 \; |\hat{u}(x)| \; |\hat{u}(y)| \, dx \, dy \right)^{1/2} \\ &= \prod_{j=1}^{n} \left(\frac{1}{(2\pi)^d} \int \int |\hat{\nu}_j(x+y)|^2 \; |\hat{u}(x)| \; |\hat{u}(-y)| \, dx \, dy \right)^{1/2}. \end{aligned}$$

Since u is real valued, we have $\hat{u}(-y) = \overline{\hat{u}(y)}$. Making this change and then making a change of variables we get the first line of (6.2). The equivalence of the first and second lines of (6.2) follows from Plancherel's Theorem. It is clear from the first line of (6.2) that $\| \cdot \|_{\gamma,2}$ is a norm.

For item 2 we first note that

$$\begin{aligned} &\int w(x,y) w(y,z) \, dm(y) && (6.13) \\ &\quad = \int_0^\infty \int_0^\infty \frac{p_{s+s'}(x,z)}{\sqrt{\pi s}\sqrt{\pi s'}} \, ds \, ds' \\ &\quad = \int_0^\infty p_t(x,z) \left(\frac{1}{\pi} \int_0^t \frac{1}{\sqrt{t-s'}\sqrt{s'}} \, ds' \right) dt \\ &\quad = \left(\frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{1-s'}\sqrt{s'}} \, ds' \right) \int_0^\infty p_t(x,z) \, dt = u(x,z). \end{aligned}$$

(It is interesting to note that w is the potential density of the process X_{T_t} where T_t is the stable subordinator of index $1/2$. In operator notation (6.13) says that $W^2 = U$ where W and U are operators with kernels w and u respectively.)

Given (6.13), the proof of 2 follows from [20, Lemma 3.2] with $h(\lambda)\,d\lambda = m(d\lambda)$. However, it is simple to complete the proof as follows: Using (6.13)

$$\begin{aligned}\prod_{j=1}^{n} u(z_j, z_{j+1}) &= \prod_{j=1}^{n} \int w(z_j, \lambda_j) w(\lambda_j, z_{j+1})\, dm(\lambda_j) \\ &= \prod_{j=1}^{n} \int w(z_j, \lambda_j) w(\lambda_{j-1}, z_j)\, dm(\lambda_j)\end{aligned} \tag{6.14}$$

in which $z_{n+1} = z_1$ and $\lambda_0 = \lambda_n$. It follows from this that

$$\begin{aligned}&\Big| \int \prod_{j=1}^{n} u(z_j, z_{j+1}) \prod_{j=1}^{n} d\nu_j(z_j) \Big| \\ &= \left| \int \prod_{j=1}^{n} \left(\int w(z_j, \lambda_j) w(\lambda_{j-1}, z_j)\, d\nu_j(z_j) \right) \prod_{j=1}^{n} dm(\lambda_j) \right| \\ &\le \prod_{j=1}^{n} \left(\int\!\!\int \left(\int w(z_j, s) w(t, z_j)\, d\nu_j(z_j) \right)^2 dm(s)\, dm(t) \right)^{1/2},\end{aligned} \tag{6.15}$$

where, for the final inequality, we use the same argument we used in (6.11).

Lastly, set

$$M_\nu(x, z) = \int w(x, y) w(y, z)\, d\nu(y). \tag{6.16}$$

Since $\|\nu\|_w$ is the L^2 norm of M_ν, and $M_{\nu+\nu'} = M_\nu + M_{\nu'}$, we see that $\|\nu\|_w$ is a norm. (This can also be viewed as the Hilbert-Schmidt norm of the operator defined by the kernel M_ν). □

Remark 6.1 Condition 2 which holds for general Markov processes is actually equivalent to 1 when the Markov process is a killed Lévy process, in which case the reference measure is Lebesgue measure. To see this we note that when $w(x, y) = w(x - y)$, $u = w * w$ so that $|\hat{u}| = |\hat{w}|^2$. Using this we see that the L^2-norm of the Fourier transform of M_ν equals $\left(\int |\hat{\nu}(x)|^2 \gamma(x)\, dx \right)^{1/2}$.

We now consider proper norms with respect to general kernels.

Theorem 6.2 *For kernels $\{u(x,y), x, y \in S\}$ satisfying the given additional conditions, the following norms are proper norms on $\mathcal{M}(S)$ with respect to u:*

4. $\|\nu\|_{u^2,\infty}$ defined in (4.25).
5. For $u \geq 0$ and positive definite, i.e., $\iint u(x,y)\,d\nu(x)\,d\nu(y) \geq 0$, for all measures $\nu \in \mathcal{M}(S)$,

$$\|\nu\|_{(2)} := \left(\int \int (u(x,y) + u(y,x))^2 \, d|\nu|(x)\, d|\nu|(y) \right)^{1/2}. \tag{6.17}$$

Proof Item 4 follows from Lemma 4.3.

To prove 5 we note that the same argument that gives (6.11) shows that

$$\begin{aligned} &\Big| \int u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y_1) \prod_{j=1}^{k} d\nu_j(y_j) \Big| \\ &\quad \leq \prod_{j=1}^{k} \left(\int \int u^2(y_j, y_{j+1})\, d|\nu_j|(y_j)\, d|\nu_{j+1}|(y_{j+1}) \right)^{1/2}. \end{aligned} \tag{6.18}$$

Furthermore, since $u(x,y)$ is positive definite, $u(x,y) + u(y,x)$ is positive definite, and therefore, by Schur's Theorem [4, p. 182], $(u(x,y) + u(y,x))^2$ is positive definite. Consequently

$$\begin{aligned} &\left(\int \int u^2(x,y)\, d|\nu|(x)\, d|\nu'|(y) \right)^{1/2} \\ &\leq \left(\int \int (u(x,y) + u(y,x))^2 \; d|\nu|(x)\, d|\nu'|(y) \right)^{1/2} \\ &\leq \|\nu\|_{(2)}^{1/2} \; \|\nu'\|_{(2)}^{1/2}, \end{aligned} \tag{6.19}$$

where, for the second inequality, we use the fact that $(u(x,y) + u(y,x))^2$ is positive definite. Using this in (6.18) gives

$$\Big| \int u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y_1) \prod_{j=1}^{k} d\nu_j(y_j) \Big| \leq \prod_{j=1}^{k} \|\nu_j\|_{(2)}. \tag{6.20}$$

We see from (1.1) that $\|\cdot\|_{(2)}$ is an L^2 norm. (It is the L^2 norm of a second order Gaussian chaos with kernel $u(x,y) + u(y,x)$.) $\square$

Remark 6.2 Let $\psi = \{\psi(\nu), \nu \in \mathcal{V}\}$ be a permanental process. It follows from Definition 1.1 that

$$E\psi^2(\nu) = \int\int u(x,y)u(y,x)\,d\nu(x)\,d\nu(y). \tag{6.21}$$

It follows from the remark in the paragraph containing (1.11) that when u is symmetric, i.e., when ψ is a second order Gaussian chaos,

$$\|\cdot\|_2 := \left(E\psi^2(\nu)\right)^{1/2} = \left(\int\int u^2(x,y)\,d\nu(x)\,d\nu(y)\right)^{1/2} \tag{6.22}$$

is a proper norm for $\mathcal{V}$ with respect to u, which implies that

$$\|\nu_x - \nu_y\|_\Xi \leq C\|\nu_x - \nu_y\|_2. \tag{6.23}$$

One way to evaluate whether the proper norms in Theorem 6.1 are good estimates is to see whether they give (6.23) when u is symmetric. Since we know that, up to a constant multiple, this is best possible when u is symmetric.

The proper norm in 1. is equivalent to (6.22) when $\hat{u}$ is real and positive because then

$$\frac{1}{(2\pi)^m}\int e^{i(x-y)q}|\hat{u}(q)|\,dq = u(y-x) = u(x,y). \tag{6.24}$$

We know that $\hat{u}$ is real when u is symmetric. It is positive when u is the potential density of a Lévy process, as one can see from the paragraph preceding Theorem 6.1.

More generally, the proper norm in 2. is also equivalent to (6.22) when u is symmetric as one can see by interchanging the order of integration and using (6.13).

The proper norm in 3. is equivalent to (6.22) when the transition densities $p_t(x,y)$ are symmetric, because in this case we can write both integrals in (6.8) as

$$\int_0^\infty\int p_{s/2}(x,u)p_{s/2}(u,y)\,dm(u)\,ds = \int_0^\infty p_s(x,y)\,ds = u(x,y). \tag{6.25}$$

The proper norms in 5. is obviously equivalent to (6.22) when the measures ν_j are positive. However, we need this to hold for general measures.

We introduce a potentially useful generalization of the notion of proper norm that enables us to obtain continuity conditions for certain permanental fields that are associated with Markov processes that need not be Lévy processes.

We say that a norm $\|\cdot\|$ on $\mathcal{M}(S)$ is a π-proper norm with respect to u, if for all $n\geq 2$, and all $\nu_1,\ldots,\nu_n\in\mathcal{M}_{\|\cdot\|}$,

$$\Big|\sum_{\pi\in\mathcal{P}_n}\int\prod_{j=1}^n u(x_j,x_{j+1})\prod_{i=1}^n d\nu_{\pi_j}(x_i)\Big|\leq n!C^n\prod_{j=1}^n\|\nu_j\|, \tag{6.26}$$

where $x_{n+1}=x_1$ and C is a finite constant. Note that (6.26), rather than the more restrictive condition (1.5), is all that is needed for the proof of Lemma 2.1; see the sentence following (2.28). Therefore, we can use Lemma 2.1 to see that (6.26) is equivalent to

$$\left|\mu\left(\prod_{j=1}^k L_\infty^{\nu_j}\right)\right|\leq n!C^n\prod_{j=1}^n\|\nu_j\|. \tag{6.27}$$

It then follows as in the proof of Theorem 2.1 that $\{\widetilde{\psi}(\nu),\nu\in\mathcal{R}_{\|\cdot\|}\}$ is an α-permanental field with kernel $u(x,y)$. Consequently, using either (2.37), after taking the limit as $\delta\to 0$, or (1.1) directly, along with (6.26), we obtain

$$\left|E\left(\prod_{j=1}^n\widetilde{\psi}(\nu_j)\right)\right|\leq n!C_\alpha^n\prod_{j=1}^n\|\nu_j\|. \tag{6.28}$$

As in the proof of Lemma 3.1 this shows that

$$\|\widetilde{\psi}(\nu)\|_\rho\leq C'_\alpha\|\nu\|. \tag{6.29}$$

The next lemma gives an example of a π-proper norm.

Lemma 6.1 *The norm*

$$\|\nu\|_{\bar{u}}:=|\nu|(S)\vee\sup_{y,z}\Big|\int u(y,x)u(x,z)\,d\nu(x)\Big|. \tag{6.30}$$

is a π-proper norm with respect to u.

(We can not show it is a proper norm.)

Proof For n even and $y_{n+1}=y_1$ we have

$$\Big|\int u(y_1,y_2)\cdots u(y_{n-1},y_n)u(y_n,y_1)\prod_{j=1}^n d\nu_j(y_j)\Big| \tag{6.31}$$

$$= \Big|\int \left(\prod_{j\,odd,1\le j\le n}\int u(y_j,y_{j+1})u(y_{j+1},y_{j+2})\,d\nu_{j+1}(y_{j+1})\right)$$

$$\prod_{j\,odd,1\le j\le n} d\nu_j(y_j)\Big|$$

$$\le \int \prod_{j\,odd,1\le j\le n}\Big|\int u(y_j,y_{j+1})u(y_{j+1},y_{j+2})\,d\nu_{j+1}(y_{j+1})\Big|$$

$$\prod_{j\,odd,1\le j\le n} d|\nu_j|(y_j)$$

$$\le \prod_{j=1}^n \|\nu_j\|_{\bar u}.$$

Therefore, for n even, by (2.10) we get

$$\left|\mu\left(\prod_{j=1}^n L_\infty^{\nu_j}\right)\right| \le (n-1)!\prod_{j=1}^n \|\nu_j\|_{\bar u}. \tag{6.32}$$

When $n=2m+1$ is odd we use the Cauchy-Schwarz Inequality together with (6.32) to get

$$\left|\mu\left(\prod_{j=1}^n L_\infty^{\nu_j}\right)\right| \tag{6.33}$$

$$\le \left|\mu\left(\prod_{j=1}^m (L_\infty^{\nu_j})^2\right)\right|^{1/2}\left|\mu\left(\prod_{j=m+1}^{2m+1}(L_\infty^{\nu_j})^2\right)\right|^{1/2}$$

$$\le n!\prod_{j=1}^n \|\nu_j\|_{\bar u}.$$

$\square$

7 Continuous additive functionals of Lévy processes

The main purpose of this section is to prove Theorem 1.4. We begin with two lemmas which follow easily from results in [15]. Because the notation in [15] is different from the notation used in Theorem 1.4 it is useful to be more explicit about the relationship between a Lévy process killed at the end of an independent exponential time and the Lévy process itself, i.e., the unkilled process. Let $Y = \{Y_t, t \in R^+\}$ be a Lévy process in R^d with characteristic exponent $\bar{\kappa}$. Let $X = \{X_t, t \in R^+\}$ be the process Y, killed at the end of an independent exponential time with mean $1/\beta$. Let κ and u denote the characteristic exponent of X and the potential density of X. Then

$$\kappa(\xi) = \beta + \bar{\kappa}(\xi) \tag{7.1}$$

and

$$\widehat{u}(\xi) = \frac{1}{\kappa(\xi)} = \frac{1}{\beta + \bar{\kappa}(\xi)}. \tag{7.2}$$

Lemma 7.1 *Let X be a Lévy process in R^d that is killed at the end of an independent exponential time, with characteristic exponent κ and potential density u, and suppose that*

$$\frac{1}{|\kappa(\xi)|^2} \le C \frac{\gamma(\xi)}{|\xi|^d}, \tag{7.3}$$

where $\gamma = |\widehat{u}| * |\widehat{u}|$. Then

$$\int |\hat{\nu}(s)||\widehat{u}(s)|\, ds \le C \int_1^\infty \frac{(\int_{|\xi| \ge x} \gamma(\xi) |\hat{\nu}(\xi)|^2 \, d\xi)^{1/2}}{x (\log 2x)^{1/2}} \, dx. \tag{7.4}$$

Proof We follow the proof of [15, Lemma 5.2] with the γ of this theorem and $|\kappa(\xi)|$ replacing the γ and $(1 + \psi(\xi))$ in [15, Lemma 5.2]. It is easy to see that the proof of [15, Lemma 5.2] goes through with these changes to prove this lemma. □

Remark 7.1 Since

$$\sup_y |U\nu(y)| \le C \int |\hat{\nu}(s)||\widehat{u}(s)|\, ds, \tag{7.5}$$

it follows from (7.4) and [2, p. 285] that ν charges no polar set. It is a conjecture of Getoor that essentially all Lévy process in R^d satisfy Hunt's hypothesis (H) which is that all semipolar sets are polar. This has been proved in many cases. See for example, [21] and [6]. In these cases the condition in Theorem 1.4, that $\nu \in \mathcal{R}^+(X)$, is superfluous.

Remark 7.2 The function $\gamma(\xi)$ plays a critical role in Theorem 1.4. We note that

$$\sup_{\xi \in R^d} \gamma(\xi) < C\|u\|_2^2, \tag{7.6}$$

for some absolute constant C.

The next lemma is a generalization of [15, Lemma 5.3].

Lemma 7.2 *If*

$$C_1\tau(|\xi|) \leq |\kappa(\xi)| \leq C_2\tau(|\xi|) \qquad \forall\, \xi \in R^d \tag{7.7}$$

and $\tau(|\xi|)$ is regularly varying at infinity, then (7.3) holds.

Proof By the assumption of regular variation, for $|\xi|$ sufficiently large,

$$\begin{aligned} \gamma(\xi) &\geq \int_{|\eta|\geq 2|\xi|} \frac{d\eta}{|\kappa(\xi-\eta)||\kappa(\eta)|} \\ &\geq \int_{|\eta|\geq 2|\xi|} \frac{d\eta}{\tau(|\eta-\xi|)\tau(|\eta|)} \\ &\geq \int_{|\eta|\geq 2|\xi|} \frac{d\eta}{\tau^2(|\eta|)} \\ &\geq C\frac{|\xi|^d}{\tau^2(|\xi|)}, \end{aligned} \tag{7.8}$$

which gives (7.3). (Since this is a lower bound it holds even if the integral on the third line is infinite!) It is clear that the constant in (7.8) can be adjusted to hold for all $\xi \in R^d$. □

The following lemma provides a key estimate in the proof of Theorem 1.4.

Lemma 7.3 *Under the hypotheses of Lemma 7.2*

$$\int |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi - \lambda_1)|\, d\lambda_1 \leq C|\widehat{u}(\xi)|\|u\|_2^2. \tag{7.9}$$

Proof Using (7.7), we can treat u as though $|\hat{u}(|\xi|)|$ is regularly varying at infinity. Consequently

$$\begin{aligned}
&\int |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi-\lambda_1)|\,d\lambda_1 \qquad (7.10)\\
&\qquad \le \int_{|\lambda_1\le|\xi|/2|} |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi-\lambda_1)|\,d\lambda_1 + \int_{|\lambda_1\ge|\xi|/2|} |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi-\lambda_1)|\,d\lambda_1\\
&\qquad \le C|\widehat{u}(\xi)| \left(\int_{|\lambda_1\le|\xi|/2|} |\widehat{u}(\lambda_1)|^2\,d\lambda_1 + \int_{|\lambda_1\ge|\xi|/2|} |\widehat{u}(\lambda_1)||\widehat{u}(\xi-\lambda_1)|\,d\lambda_1 \right)\\
&\qquad \le C|\widehat{u}(\xi)| \left(\|u\|_2^2 + \gamma(\xi) \right),
\end{aligned}$$

which implies (7.9). □

Proof of Theorem 1.4 This theorem is an immediate consequence of Theorem 5.2. We begin by showing that Theorem 5.2 (ii) holds. We take $\phi(dx) = \rho(dx) = e^{-|x|^2/2}\,dx$ and we set $h(y,x) = h(x-y) = u_1(x-y)$ where $u_1(y) = \int_1^\infty p_t(y)\,dt$. We have

$$|\hat{h}(\lambda)| = |\widehat{u}(\lambda)|\,e^{-Re\,\kappa(\lambda)}. \tag{7.11}$$

To show that condition (i) of Lemma 5.1 holds we show that

$$\sup_{x,y} \left| \int u(y,z)h(z,x)\,d\nu(z) \right| \le C \int |\hat{\nu}(s)||\widehat{u}(s)|\,ds, \tag{7.12}$$

and

$$\sup_{x,y} \left| \int u(y,z) \left(\int u(z,w)h(w,x)\,d\phi(w) \right) \,d\nu(z) \right| \le C \int |\hat{\nu}(s)||\widehat{u}(s)|\,ds. \tag{7.13}$$

When (1.20) holds, it follows from (7.4) that the right-hand side is finite. Therefore, replacing ν in (7.12) and (7.13) by $\nu_r - \nu_{r'}$, so that $|\hat{\nu}(s)|$ is replaced by $|e^{ir\cdot s} - e^{ir'\cdot s}|\,|\hat{\nu}(s)|$, we see that condition (i) of Lemma 5.1 follows from (7.4) and the Dominated Convergence Theorem.

To obtain (7.12) we write

$$\begin{aligned}
&\int u(y,z)h(z,x)\,d\nu(z) \qquad (7.14)\\
&\quad = \int u(z-y)h(x-z)\,d\nu(z)\\
&\quad = \int e^{i(z-y)\lambda_1}\widehat{u}(\lambda_1)e^{i(x-z)\lambda_2}\hat{h}(\lambda_2)\,d\lambda_1\,d\lambda_2\,d\nu(z)\\
&\quad = \int \hat{\nu}(\lambda_1-\lambda_2)e^{-iy\lambda_1}\widehat{u}(\lambda_1)e^{ix\lambda_2}\hat{h}(\lambda_2)\,d\lambda_1\,d\lambda_2.
\end{aligned}$$

Hence

$$\sup_{x,y}\left|\int u(y,z)h(z,x)\,d\nu(z)\right| \le \int |\hat{\nu}(s)| \left(\int |\widehat{u}(s+\lambda_2)||\hat{h}(\lambda_2)|\,d\lambda_2\right)\,ds. \tag{7.15}$$

We complete the proof of (7.12) by showing that

$$\int |\widehat{u}(s+\lambda)||\hat{h}(\lambda)|\,d\lambda \le C|\widehat{u}(s)|. \tag{7.16}$$

We have

$$\int |\widehat{u}(s+\lambda)||\hat{h}(\lambda)|\,d\lambda = C\int \frac{e^{-Re\,\kappa(\lambda)}}{|\kappa(s+\lambda)||\kappa(\lambda)|}\,d\lambda. \tag{7.17}$$

Using the same inequalities used in the proof of Lemma 7.2, we see that

$$\int_{|\lambda|\le|s|/2} \frac{e^{-Re\,\kappa(\lambda)}}{|\kappa(s+\lambda)||\kappa(\lambda)|}\,d\lambda \le C\frac{1}{|\kappa(s)|}\int \frac{e^{-Re\,\kappa(\lambda)}}{|\kappa(\lambda)|}\,d\lambda. \tag{7.18}$$

and

$$\int_{|\lambda|\ge|s|/2} \frac{e^{-Re\,\kappa(\lambda)}}{|\kappa(s+\lambda)||\kappa(\lambda)|}\,d\lambda \le C\frac{1}{|\kappa(s)|}\int e^{-Re\,\kappa(\lambda)}\,d\lambda. \tag{7.19}$$

Using (1.16), (7.18) and (7.19) in (7.17) and then (1.19) we get (7.16).

In a similar manner to how we obtained (7.15) by taking Fourier transforms, we see that

$$\begin{aligned}
&\sup_{x,y}\left|\int u(y,z)\left(\int u(z,w)h(w,x)\,d\phi(w)\right)\,d\nu(z)\right| \qquad (7.20)\\
&\quad\le \int\left(\int |\hat{\phi}(\lambda_1-\lambda_2)||\hat{h}(\lambda_2)|\,d\lambda_2\right)|\widehat{u}(\lambda_1)||\widehat{u}(\lambda_3)||\hat{\nu}((\lambda_1+\lambda_3))|\,d\lambda_1\,d\lambda_3\\
&\quad= \int\left(\int |\hat{\phi}(\lambda_1-\lambda_2)||\hat{h}(\lambda_2)|\,d\lambda_2\right)|\widehat{u}(\lambda_1)||\widehat{u}(s-\lambda_1)|\,d\lambda_1|\hat{\nu}(s)|\,ds.
\end{aligned}$$

Clearly, since $\phi = e^{-|x|^2/2}\,dx$, $|\hat{\phi}(\lambda)| \le C|\widehat{u}(\lambda)|$. Therefore, by (7.16)

$$\int |\hat{\phi}(\lambda_1-\lambda_2)||\hat{h}(\lambda_2)|\,d\lambda_2 \le C|\widehat{u}(\lambda_1)|. \tag{7.21}$$

Using this (7.20) and Lemma 7.3 we get (7.13).

We now show that condition (ii) of Lemma 5.1 holds. We have already seen that $\nu \in \mathcal{R}^+$. By Theorem 6.1, $\|\cdot\|_{\gamma,2}$ is a proper norm for u, and it follows from (1.20) that $\{\nu_x, x \in R^d\} \subseteq \mathcal{R}^+_{\|\cdot\|_{\gamma.2}}$. Therefore it follows from

Theorem 2.1 that there exists a permanental field $\psi = \{\psi(\nu_x), x \in R^d\}$ with kernel u. To complete the proof of continuity part of Theorem 1.4 we need the following lemma which is proved below.

Lemma 7.4 *The permanental field $\{\psi(\nu_x), x \in R^d)$ is continuous on $(R^d, |\cdot|)$ almost surely. Furthermore, for any compact set $D \in R^d$*

$$\lim_{\delta\to 0} E\left(\sup_{\substack{|x-y|\le\delta\\ x,y\in D}}\Big|\psi(\nu_x)-\psi(\nu_y)\Big|^2\right)=0. \tag{7.22}$$

Proof of Theorem 1.4, continued We use (7.22) and Remark 4.1, to see that condition (ii) of Lemma 5.1 holds with the metric d being the Euclidean metric on R^d. Therefore, the conditions in Theorem 5.2 (ii) hold and since d is the Euclidean metric the condition in Theorem 5.2 (i) also holds. The continuity portion of Thereom 1.4 now follows from Theorem 5.2.

Proof of Lemma 7.4 This lemma follows easily from the proof of [15, Theorem 1.6] with the γ of this theorem replacing the γ in [15, Theorem 1.6]. The gist of the proof of [15, Theorem 1.6] is that (1.20) implies that for compact sets D of R^d

$$\lim_{\delta\to 0} J_{\overline{\mathcal{V}},\|\cdot\|_{\gamma,2},\lambda}(\delta)=0, \tag{7.23}$$

where $\overline{\mathcal{V}}=\{\nu_x, x\in D\}$ and λ is Lebesgue measure on R^d. (See Section 3 for notation.)

By Theorem 1.2 we get that $\psi(\nu_x)$ is continuous on $(\overline{\mathcal{V}}, \|\cdot\|_{\gamma,2})$ almost surely and (7.22) holds with $|x-y|$ replaced by $\|\nu_x-\nu_y\|_{\gamma,2}$ and $x,y\in D$ replaced by $\nu_x,\nu_y\in\overline{\mathcal{V}}$. Since

$$\|\nu_x-\nu_{x+h}\|_{\gamma,2}=C\left(\int_{\xi\in R^d}\sin^2\frac{\xi h}{2}\gamma(\xi)|\hat\nu(\xi)|^2\,d\xi\right)^{1/2} \tag{7.24}$$

we see that $\psi(\nu_x)$ is continuous on R^d and we get (7.22) as stated. □

Strengthening the hypotheses of Theorem 1.4 we get the simple estimate of $\gamma(\xi)$ in the next lemma.

Lemma 7.5 *Under the hypotheses of Lemma 7.2 assume also that τ is regularly varying at infinity with index greater than $d/2$ and less than d. Then*

$$\gamma(\xi)\le C\frac{|\xi|^d}{\tau^2(|\xi|)} \tag{7.25}$$

for all $|\xi|$ sufficiently large.

Proof By (7.7)

$$\gamma(\xi) \leq C\int_{R^d} \frac{d\eta}{\tau(|\xi-\eta|)\tau(|\eta|)} \tag{7.26}$$
$$= C\left(\int_{|\eta|\leq|\xi|/2} + \int_{|\xi|/2\leq|\eta|\leq(3/2)|\xi|} + \int_{|\eta|\geq 3/2|\xi|}\right)\cdots = I + II + III.$$

In bounding (7.26) we can take $\tau(|\cdot|)$ to be increasing. Therefore for $|\eta| \leq |\xi|/2$, $\tau(|\xi-\eta|) \geq \tau(|\xi/2|) \geq c\tau(|\xi|)$ for some constant c and all $|\eta|$ sufficiently large. Consequently

$$I \leq C_1\frac{1}{\tau(|\xi|)}\int_{|\eta|\leq|\xi|/2}\frac{|\eta|^{d-1}d|\eta|}{\tau(|\eta|)} \leq C_1'\frac{|\xi|^d}{\tau^2(|\xi|)}, \tag{7.27}$$

because τ is regularly varying at infinity with index less than d.

When $|\eta| \geq 3|\xi|/2$, $\tau(|\xi-\eta|) \geq \tau(|\eta/3|) \geq c'\tau(|\eta|)$ for some constant c' and all $|\eta|$ sufficiently large. Therefore,

$$III \leq C_3\int_{|\eta|\geq 3|\xi|/2}\frac{|\eta|^{d-1}d|\eta|}{\tau(|\eta|)\tau(|\eta|)} \leq C_3'\frac{|\xi|^d}{\tau^2(|\xi|)}, \tag{7.28}$$

because τ is regularly varying at infinity with index greater than $d/2$. Finally

$$II \leq C_2\frac{1}{\tau(|\xi|)}\int_{|\xi|/2\leq|\eta|\leq 3|\xi|/2}\frac{d\eta}{\tau(|\xi-\eta|)} \tag{7.29}$$
$$\leq C_2'\frac{1}{\tau(|\xi|)}\int_{0\leq|\eta|\leq 3|\xi|}\frac{d\eta}{\tau(|\eta|)} \leq C_2''\frac{1}{\tau(|\xi|)}\int_{0\leq|\eta|\leq 3|\xi|}\frac{|\eta|^{d-1}d|\eta|}{\tau(|\eta|)}$$
$$\leq C_2'''\frac{|\xi|^d}{\tau^2(|\xi|)},$$

as in (7.27). □

Remark 7.3 We give some details on how the examples in Example 1.1, 1. and 2. are obtained.

1. It is easy to see that (1.24) follows from (1.20) and Lemma 7.5.
2. In this case the estimate in (7.25) is not correct. To find a bound for $\gamma(\xi)$ we look at the proof of Lemma 7.5 with $d = 2$ and τ as given in

(1.25). The bounds in III remains the same but the bounds in I and II are now

$$C\frac{|\xi|^2 \log |\xi|}{\tau^2(|\xi|)} \tag{7.30}$$

for all $|\xi|$ sufficiently large. Given this the rest of the argument is essentially the same as in 1.

We take up the proof of the modulus of continuity assertion in Theorem 1.4 in the next subsection.

7.1 Moduli of continuity

We begin with a modulus of continuity result for certain permanental processes, including the those considered in Theorem 1.4.

Theorem 7.1 *Let $\{\psi(\nu), \nu \in \mathcal{V}\}$ be a permanental process with kernel u, where $\mathcal{V} = \{\nu_x, x \in R^n\}$ is a family of measures such that*

$$\|\nu_x - \nu_y\| \le \varrho(|x-y|), \tag{7.31}$$

where $\|\cdot\|$ is a proper norm on $\mathcal{V}$ with respect to u, and ϱ is a strictly increasing function. Let

$$\omega(\delta) = \varrho(\delta) \log 1/\delta + \int_0^\delta \frac{\varrho(u)}{u}\, du, \tag{7.32}$$

and assume that the integral is finite. Then for each $K > 0$ there exists a constant C such that

$$\limsup_{\delta \to 0} \sup_{\substack{|x-y| \le \delta \\ x,y \in [-K,K]^n}} \frac{\psi(\nu_x) - \psi(\nu_y)}{\omega(\delta)} \le C \qquad a.s. \tag{7.33}$$

In particular if ϱ is a regularly varying function at zero with index greater than zero, we can take

$$\omega(\delta) = \varrho(\delta) \log 1/\delta. \tag{7.34}$$

Proof This is proved in [17, Section 7.2] in a slightly different setting. For it to hold in our setting just change $(\log 1/u)^{1/2}$ in [17, (7.90)] to $\log 1/u$ and continue the proof with this change. This takes into account the fact that in (3.2) we have a log rather than $(\log)^{1/2}$, which is what we have when dealing with Gaussian processes. $\square$

Example 7.1 We consider Theorem 7.1 in the case where $u(x,y)=u(y-x)$ and the proper norm is $\|\cdot\|_{\gamma,2}$. By (6.2)

$$\begin{aligned}\|\nu_x-\nu_y\|_{\gamma,2} &\leq C \left(\int |\hat{\nu}_x(\lambda)-\nu_y(\lambda)|^2\,\gamma(\lambda)\,d\lambda\right)^{1/2} \\ &\leq C\left(\int \sin^2\frac{(x-y)\lambda}{2}|\hat{\nu}(\lambda)|^2\,\gamma(\lambda)\,d\lambda\right)^{1/2} \\ &\leq C\,\varphi(|x-y|),\end{aligned} \tag{7.35}$$

where φ is given in (1.23). Note that if (1.20) holds then $\int(\varphi(u)/u)\,du<\infty$. Therefore, if (1.20) holds, the results in (7.33)–(7.34) hold with ϱ replaced by φ.

Proof of Theorem 1.4, modulus of continuity This follows from Theorem 7.1, Example 7.1 and the second isomorphism theorem, Theorem 4.2, as in the proof of a similar result in[16, Section 7]. Note that the requirement that $U^1\mu<\infty$ in [16, Theorem 2.2] follows from (1.20), (7.4) and (7.5). $\square$

Remark 7.4 The results in Example 1.1, 3 and 4 come from (7.35) and an estimate of φ as given in (1.23).

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Yves Le Jan
Equipe Probabilités et Statistiques
Université Paris-Sud, Bâtiment 425
91405 Orsay Cedex
France
yves.lejan@math.u-psud.fr

Michael Marcus
Department of Mathematics
City College, CUNY
New York, NY 10031
U.S.A.
mbmarcus@optonline.net

Jay Rosen
Department of Mathematics
College of Staten Island, CUNY
Staten Island, NY 10314
U.S.A.
jrosen30@optimum.net